THE DENDROIDAL CATEGORY IS A TEST CATEGORY

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ABSTRACT. We prove that the category of trees Ω is a test category in the sense of Grothendieck. This implies that the category of dendroidal sets is endowed with the structure of a model category Quillen-equivalent to spaces. We show that this model category structure, up to a change of cofibrations, can be obtained as an explicit left Bousfield localisation of the operadic model category structure.

INTRODUCTION

The notion of a test category was introduced by Grothendieck in his influential manuscript [8], with the aim of axiomatising those small categories which could play a role similar to the category Δ of simplices and serve as building blocks to describe all homotopy types of spaces. The theory of test categories has been described and further developed in [11], [3], [10], [12], [4] and [1]. Here one can find a list of examples of test categories, which includes, in addition to Δ, familiar categories such as the category of cubes parametrising cubical sets, and Joyal's category Θ_n parametrising a notion of n-dimensional category. The goal of this paper is two-fold. First of all, we wish to add some further examples to this list, by showing that the category Ω of trees which parametrises dendroidal sets is a test category. The argument will also show that variations of Ω such as similar categories of planar trees and closed trees are test categories. It follows that there is a Quillen model structure on the category of dendroidal sets which models the homotopy category of spaces. The second goal of this paper is to explain the relation of this model structure to the model structure that the category of dendroidal sets was originally designed for, namely the so-called operadic model structure which models the homotopy theory of topological (or simplicial) coloured operads. Indeed, we will show that up to a small change in the class of cofibrations only, the first model structure coming from the fact that Ω is a test category can be obtained as a left Bousfield localisation of the second, operadic model structure.

The plan of our paper is as follows. In the first section, we will review the basic definitions of the theory of test categories. In Section 2, we will present a proof of the fact that the simplex category is a test category which is somewhat different from the ones occurring in the literature, and is based on the fact that the product of two simplices can be written as a union of other simplices indexed by shuffles. The proof that Ω is a test category will be broken up into two parts. The first part shows the fact that the classifying space of Ω is contractible. This fact has been known for quite some time, but a proof has never been published. The second part of the proof will again use shuffles and follows the same pattern as the argument for simplices. In the final section, we discuss the relation to the operadic model structure on dendroidal sets mentioned above.

The results of this paper go back quite a while, and were presented in 2013 at the conference celebrating the 65th birthday of G. Maltsiniotis. We would like to thank G. Maltsiniotis for encouraging us to write up the results, and apologise for the fact that it has taken us a while.

We are grateful to the referee for carefully reading the paper.

1. Preliminaries on test categories

In this section, we review some basics of the theory of test categories introduced by Grothendieck in [8]. For more detailed expositions and proofs, we refer the reader to [11] and [3].

We begin with a bit of notation and terminology.

1.1. We denote by $N : \mathcal{C}at \to \widehat{\Delta}$ the nerve functor from small categories to simplicial sets. A functor $u : A \to B$ between small categories is said to be a *weak equivalence* if its nerve $N(u)$ is a weak homotopy equivalence of simplicial sets. We say that a small category A is *aspherical* (or *weakly contractible*) if the unique functor from A to the terminal category is a weak equivalence.

1.2. Let $u : A \to B$ be a functor. If b is an object of B, we denote by A/b the category $A \times_B (B/b)$, where B/b is the category of objects over b. This category is sometimes denoted by $u \downarrow b$. If F is a presheaf on A and u is the Yoneda embedding, the category A/F is the category of elements of F.

We now introduce the basic definitions of the theory of test categories.

1.3. Let A be a small category. We have a pair of adjoint functors

$$i_A : \quad \begin{matrix} \widehat{A} & \to & \mathcal{C}at, \\ F & \mapsto & A/F \end{matrix} \qquad i_A^* : \quad \begin{matrix} \mathcal{C}at & \to & \widehat{A} \\ C & \mapsto & \big(a \mapsto \mathsf{Hom}_{\mathcal{C}at}(A/a, C)\big) \end{matrix}$$

between presheaves on A and small categories. A morphism $f : X \to Y$ of presheaves on A is said to be a *weak equivalence* if $i_A(f) : A/X \to A/Y$ is a weak equivalence of categories. A presheaf X on A is *aspherical* if the category A/X is aspherical.

Definition 1.4. Let A be a small category.
  (a) The category A is said to be a *weak test category* if, for every small category C, the counit functor $\varepsilon_C : i_A i_A^* C \to C$ is a weak equivalence.
  (b) The category A is said to be a *local test category* if, for every object a of A, the slice category A/a is a weak test category.
  (c) The category A is said to be a *test category* if it is both a weak test category and a local test category.

The following proposition shows that to understand test categories, it is enough to understand local test categories:

Proposition 1.5 (Grothendieck). *A small category A is a test category if and only if the following two conditions hold:*
  (a) *A is aspherical;*
  (b) *A is a local test category.*

Proof. See [11, Remark 1.5.4]. □

We now move on to a characterisation of local test categories in terms of intervals.

1.6. An *interval* of a presheaf category $\widehat{A}$ consists of a presheaf I endowed with two global sections $\partial_0, \partial_1 : * \to I$. The interval is said to be *separating* if the induced map $\partial_0 \amalg \partial_1 : * \amalg * \to I$ is a monomorphism.

For instance, the subobject classifier L_A of the topos $\widehat{A}$, also called the *Lawvere object* of $\widehat{A}$, is canonically endowed with the structure of a separating interval, ∂_0 and ∂_1 corresponding respectively to the empty subobject and the maximal subobject of $*$.

We say that an interval I is *locally aspherical* if, for every presheaf X on A, the projection map $X \times I \to X$ is a weak equivalence. A direct application of Quillen's Theorem A shows that it is enough to require this property when X is representable.

Theorem 1.7 (Grothendieck)**.** *Let A be a small category. The following conditions are equivalent:*

    (a) *A is a local test category;*
    (b) *L_A is locally aspherical;*
    (c) *there exists a locally aspherical separating interval in $\widehat{A}$.*

Proof. See [11, Theorem 1.5.6]. $\square$

We moreover have from [3] the following characterisation of local test categories in terms of model categories:

Theorem 1.8. *Let A be a small category. The following conditions are equivalent:*

    (a) *A is a local test category;*
    (b) *there exists a model category structure on $\widehat{A}$ whose cofibrations are the monomorphisms and whose weak equivalences are the weak equivalences of presheaves as defined in 1.3.*

Moreover, if these conditions are fulfilled, the model category structure of the second condition is combinatorial and proper.

Proof. See [3, Corollary 4.2.18] for the implication (a) $\Rightarrow$ (b) (and the fact that the model category structure is combinatorial) and [3, Theorem 4.1.19] for the reciprocal. The properness under these assumptions follows from [3, Corollary 4.2.19 and Example 4.3.22]. $\square$

We now characterise test categories in terms of model categories.

1.9. Let A be a small category. Denote by

$$\lambda_! = N i_A : \widehat{A} \to \widehat{\Delta}$$

the composition of i_A with the nerve functor. The functor $\lambda_!$ preserves colimits (see for instance [3, Corollary 3.2.10]) and hence admits a right adjoint λ^*. We thus have an adjoint pair

$$\lambda_! : \widehat{A} \rightleftarrows \widehat{\Delta} : \lambda^*.$$

The functor $\lambda_!$ also preserves pullbacks and hence monomorphisms.

Theorem 1.10. *Let A be a small category. The following conditions are equivalent:*

    (a) *A is a test category;*

(b) *there exists a model category structure on $\widehat{A}$ whose cofibrations are the monomorphisms and for which the adjoint pair*

$$\lambda_! : \widehat{A} \rightleftarrows \widehat{\Delta} : \lambda^*$$

is a Quillen equivalence, where the category $\widehat{\Delta}$ of simplicial sets is endowed with the Kan–Quillen model category structure.

Proof. The implication (a) $\Rightarrow$ (b) is a consequence of [3, Proposition 4.2.26 and Remark 4.2.27]. Let us prove the converse. Since every object of $\widehat{A}$ is cofibrant, the left Quillen functor $\lambda_!$ preserves and detects weak equivalences. This shows that the weak equivalences of the model category structure on $\widehat{A}$ are the weak equivalences of presheaves as defined in 1.3. In particular, by Theorem 1.8, A is a local test category. By Proposition 1.5, to conclude the proof, it suffices to show that A is aspherical. But since Δ_0 is a fibrant simplicial set, the morphism $\lambda_!\lambda^*(\Delta_0) \to \Delta_0$ is a weak equivalence, and since $\lambda_!\lambda^*(\Delta_0) \simeq N(A)$ we get the result. $\qquad\square$

Remark 1.11. We saw in the proof that the model category structure of (b) is actually unique (and in particular coincides with the one of Theorem 1.8).

2. A proof that Δ is a test category

In this section, we give a new proof of the fact that the simplex category Δ is a test category. The proof for Ω will follow a similar pattern.

2.1. To prove that Δ is a test category, it suffices to show that for any $n \geq 0$, the simplicial set $\Delta_1 \times \Delta_n$ is aspherical, where Δ_m denotes the standard m-simplex. Indeed, since Δ has a terminal object, it is aspherical. Moreover, the object Δ_1 is clearly a separating interval and our claim thus follows from Proposition 1.5 and Theorem 1.7.

From now on, we fix $m, n \geq 0$. We will prove more generally that $\Delta_m \times \Delta_n$ is aspherical. This fact is well-known and follows from classical results but we will give an elementary proof in the spirit of the theory of test categories.

2.2. Recall that the simplicial set $\Delta_m \times \Delta_n$ can be written as a union of subsimplicial sets

$$\Delta_m \times \Delta_n = \bigcup_{\sigma \in \mathrm{Sh}_{m,n}} F_\sigma,$$

where $\mathrm{Sh}_{m,n}$ is the set of (m,n)-shuffles and each F_σ is isomorphic to Δ_{m+n} (see for instance [7, Chapter II, Section 5]). The only additional fact we will need about these F_σ's is that they all contain the vertex $(0,0)$ of $\Delta_m \times \Delta_n$.

To prove that $\Delta_m \times \Delta_n$ is aspherical, we will use the following general lemma:

Lemma 2.3. *Let A be a small category and let F be a presheaf on A. Suppose that F can be written as a non-empty finite union of subpresheaves*

$$F = \bigcup_{i \in I} F_i$$

satisfying the following condition: for every non-empty $J \subseteq I$, the intersection presheaf

$$F_J = \bigcap_{j \in J} F_j$$

is aspherical. Then F is aspherical.

Proof. The case of a binary union is [11, Proposition 1.2.7]. The finite case follows by induction. □

Proposition 2.4. *The simplicial set $\Delta_m \times \Delta_n$ is aspherical.*

Proof. Let J be a non-empty set of (m,n)-shuffles. By Lemma 2.3, it suffices to show that

$$F_J = \bigcap_{j \in J} F_j \subset \Delta_m \times \Delta_n$$

is aspherical. We will show that F_J is actually a representable presheaf. Indeed, each of the F_j's is the nerve of a subposet of the poset associated to $\Delta_m \times \Delta_n$ and the intersection F_J is the nerve of the intersection of these subposets. The result follows from the fact that this intersection subposet is a non-empty (as it contains $(0,0)$) finite linear order. □

Corollary 2.5. *The simplex category Δ is a test category.*

*Proof.* This follows from the previous proposition (see 2.1). □

Remark 2.6. The proof actually shows that Δ is a *strict* test category (see [11, Section 1.6]).

3. The tree category Ω

The tree category Ω was introduced by the third author and Weiss in [13]. The purpose of this section is to recall some of the main definitions related to Ω.

3.1. By an *operad*, we will always mean a symmetric coloured operad. We will denote by *Oper* the category of operads. If P is an operad and $c_1, \ldots, c_n, d$ are colours, we will denote by $P(c_1, \ldots, c_n; d)$ the set of operations in P from $c_1, \ldots, c_n$ to d.

3.2. Similarly, by a *tree*, we will always mean a finite non-planar rooted tree. Here is an example of such a tree:
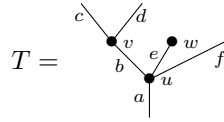
Every tree T generates a coloured operad $\Omega(T)$ in the following way. Choose a planar structure on T and consider the non-symmetric coloured collection $\Omega_0(T)$ whose colours are the edges of T and whose operations are given by the vertices of T (the planar structure fixes the source of such an operation). The operad $\Omega(T)$ is then the free coloured operad on $\Omega_0(T)$. It does not depend on the choice of the planar structure.

We will denote by η the tree with one edge and no vertices, and, for $n \geq 0$, by C_n the n-corolla, that is, the tree with one vertex and n leaves.

3.3. The category Ω is defined in the following way. Its objects are trees, and if S and T are two trees, a morphism $S \to T$ in Ω is given by a morphism of operads $\Omega(S) \to \Omega(T)$. A presheaf on Ω is called a *dendroidal set*. We will consider the Yoneda embedding $\Omega \hookrightarrow \widehat{\Omega}$ as an inclusion, thus identifying each object of Ω with its associated dendroidal set.

3.4. There is a fully faithful functor $i : \Delta \to \Omega$ defined by

$$\Delta_n \mapsto L_n = \begin{array}{c} \bullet^{\,0} \\ \bullet^{\,1} \\ \vdots \\ \bullet^{\,n-1} \\ \bullet^{\,n} \end{array}$$

We will consider i as an inclusion and we will thus identify Δ_n with L_n. In particular, Δ_0 will be identified with η. The image of i being a sieve, the left Kan extension $i_! : \widehat{\Delta} \to \widehat{\Omega}$ sends a simplicial set X to the dendroidal set obtained by extending X by $\varnothing$ at trees not in the image of i. This functor is fully faithful.

3.5. Every map of Ω factors as a *degeneracy* followed by an isomorphism followed by a *face map*. The face maps are generated by elementary faces. An *elementary face* is either an inner face or an outer face defined as follows. Let T be a tree. For e an inner edge of T (that is an edge between two vertices), define T/e to be the tree obtained by contracting e. There is a map $\partial_e : T/e \to T$ in Ω corresponding to the composition in $\Omega(T)$ of the two operations associated to the end-vertices of e. Such a map is called an *inner face*. If T has at least two vertices and v is a vertex with exactly one adjacent inner edge, define T/v to be the tree obtained by chopping off v. There is a map $\partial_v : T/v \to T$ corresponding to the obvious inclusion of operads. If T has exactly one vertex, that is, if T is a corolla, there is one map $\eta \to T$ for each edge of T. These two kinds of maps are called *outer faces*.

Similarly, degeneracies are generated by elementary degeneracies defined as follows. If T is a tree, for each edge e there is an *elementary degeneracy* $\sigma_e : S \to T$, where S is obtained from T by inserting a vertex in the middle of e and σ_e corresponds to the identity of e in the operad $\Omega(T)$.

For more on faces and degeneracies, see [13, Section 3].

3.6. For a tree T and an inner edge e of T, we denote by Λ_T^e the maximal subobject of T in $\widehat{\Omega}$ not containing the face $\partial_e : T/e \to T$. The inclusions of the form $\Lambda_T^e \hookrightarrow T$ are called *inner horn inclusions*. A dendroidal set is an ∞-*operad* if it has the extension property with respect to every inner horn inclusion.

3.7. A monomorphism of dendroidal sets $X \hookrightarrow Y$ is said to be *normal* if, for any tree T, the action of the group of automorphisms of T in Ω on $Y(T) \backslash X(T)$ is free. The class of normal monomorphisms can be characterised as the saturation (i.e., the closure under pushout, transfinite composition and retracts) of the set $\{ \partial T \hookrightarrow T \mid T \in \Omega \}$, where ∂T denotes the maximal proper subdendroidal set of T.

A dendroidal set X is said to be *normal* if the monomorphism $\varnothing \to X$ is normal. For instance, trees are normal dendroidal sets. We will need the following two facts about normal dendroidal sets: if $f : X \to Y$ is a map of dendroidal sets with Y normal, then X is normal; if moreover f is a monomorphism, then f is a normal monomorphism.

We can now formulate one of the main results of [5]:

Theorem 3.8. *There exists a combinatorial model category structure on $\widehat{\Omega}$ whose cofibrations are the normal monomorphisms and whose fibrant objects are the ∞-operads.*

This model category structure will be called the *operadic model category structure* and its weak equivalences the *operadic weak equivalences*. These operadic weak equivalences can be characterised in terms of Segal core inclusions that we now define.

3.9. Let T be a tree. For each vertex v with n input edges, there is a map $C_n \to T$ corresponding to the operation associated to v in $\Omega(T)$. The *Segal core* $\mathrm{Sc}(T)$ of T is the smallest subdendroidal set of T containing the images of these maps. Inclusions of the form $\mathrm{Sc}(T) \hookrightarrow T$ are called *Segal core inclusions*.

The following characterisation of operadic weak equivalences follows from [5, Corollary 6.11] and [6, Proposition 2.6]:

Theorem 3.10. *The class of operadic weak equivalences is the smallest class W satisfying the following properties:*

(a) *W satisfies the 2-out-of-3 property;*
(b) *W contains the class of maps having the right lifting property with respect to normal monomorphisms;*
(c) *the class of normal monomorphisms which are in W is closed under pushout, transfinite composition and retracts.*
(d) *W contains the set of Segal core inclusions.*

Recall finally that dendroidal sets are endowed with a tensor product.

3.11. We will denote by $\otimes_{\mathrm{BV}}$ the Boardman–Vogt tensor product of operads (see [2, Definition 2.14]). If X and Y are two dendroidal sets, their *tensor product* is defined by the formula

$$X \otimes Y = \varinjlim_{S \to X,\, T \to Y} N_d(\Omega(S) \otimes_{\mathrm{BV}} \Omega(T)),$$

where S and T vary among trees and N_d denotes the dendroidal nerve functor (see [13, Example 4.2]). This tensor product is symmetric but only associative up to weak equivalence (see [9, Section 6.3]). It admits η as a unit. Moreover, it preserves colimits in each variable.

4. THE CATEGORY Ω IS ASPHERICAL

The goal of this section is to show that the category Ω is aspherical.

4.1. The *décalage* $D(T)$ of a tree T is the tree

$$D(T) = T \amalg_\eta C_1,$$

where $\eta \to C_1$ is the map corresponding to the unique leaf of C_1 and $\eta \to T$ is the root map of T, that is, the map corresponding to the root of T. For each tree T, we have a diagram

$$T \xrightarrow{u_T} D(T) \xleftarrow{a_T} \eta,$$

where $u_T : T \to D(T) = T \amalg_\eta C_1$ is the canonical map and $a_T : \eta \to D(T)$ is the root map of $D(T)$. In other words, $D(T)$ is obtained from T by adding a new unary vertex v_T at the root, with a new root edge a_T coming out of it:

The map $u_T : T \to D(T)$ defined above is the outer face map ∂_{v_T}.

Remark 4.2. It is tempting to think at this point that we have a zigzag of natural maps

$$T \xrightarrow{u_T} D(T) \xleftarrow{a_T} \eta$$

and hence that Ω is aspherical. Notice though that we have not defined the action of D on the morphisms of Ω. It turns out that there is no way to make D into a functor $D : \Omega \to \Omega$ for which the maps

$$T \xrightarrow{u_T} D(T) \xleftarrow{a_T} \eta$$

are natural. Indeed, consider the face map $\partial_v : S \to T$:

Clearly, this map cannot be extended to a root-preserving map

for v_S would have to be sent to a unary operation in $\Omega(D(T))$ from c to a_T and there is no such operation. Note that if there were a (nullary) vertex above b in T, then there would exist such an operation. This leads to the following definition.

Definition 4.3. A tree T is said to be *closed* if it has no leaves, that is, if there is a vertex above every edge of T. The full subcategory of Ω consisting of closed trees is denoted $\overline{\Omega}$.

4.4. The *closure* of a tree T is the tree $\mathrm{cl}(T) = \overline{T}$ obtained from T by adjoining a (nullary) vertex v_l on top of each leaf l of T. We will denote by $\eta_T : T \hookrightarrow \overline{T}$ the obvious inclusion.

If $f : S \to T$ is a map in Ω, we define $\mathrm{cl}(f) = \overline{f} : \overline{S} \to \overline{T}$ to be the unique map $\overline{S} \to \overline{T}$ extending $S \to T$, in the sense that the diagram

$$
\begin{array}{ccc}
\overline{S} & \xrightarrow{\overline{f}} & \overline{T} \\
\eta_S \uparrow & & \uparrow \eta_T \\
S & \xrightarrow{f} & T
\end{array}
$$

commutes. Since this property determines the action of $\overline{f}$ on the edges of $\overline{S}$, there is at most one such map. Its existence follows from the fact that $\Omega(\overline{T})$ has a (unique) nullary operation for each of its colours.

One checks that this defines a functor $\mathrm{cl} : \Omega \to \overline{\Omega}$ from trees to closed trees.

Example 4.5. Consider the following external face map ∂_w:

$$R = \quad\vcenter{\hbox{tree}} \qquad \longrightarrow \qquad T = \quad\vcenter{\hbox{tree}}$$

The closure $\mathrm{cl}(\partial_w)$ of ∂_w

$$\overline{R} = \quad\vcenter{\hbox{tree}} \qquad \longrightarrow \qquad \overline{T} = \quad\vcenter{\hbox{tree}}$$

is the composition of the three inner face maps ∂_d, ∂_e and ∂_f.

Proposition 4.6. *The inclusion $i : \overline{\Omega} \hookrightarrow \Omega$ admits the functor $\mathrm{cl} : \Omega \to \overline{\Omega}$ as a left adjoint.*

Proof. By the previous paragraph, we have a natural transformation $\eta : 1_\Omega \to i\,\mathrm{cl}$. Denote by $\varepsilon : \mathrm{cl}\,i = 1_{\overline{\Omega}} \to 1_{\overline{\Omega}}$ the identity natural transformation. We claim that η and ε are the unit and counit of the announced adjunction. Using the fact that ε is the identity and that η is the identity on $\overline{\Omega}$, the triangular identities reduce to the equality $\mathrm{cl}\,\eta = 1_{\mathrm{cl}}$; that is, to the fact that if T is a tree, we have $\mathrm{cl}(T \to \mathrm{cl}(T)) = 1_{\mathrm{cl}(T)}$. This is readily checked. $\square$

4.7. We now define a functor $D : \overline{\Omega} \to \overline{\Omega}$ extending the assignment $T \mapsto D(T)$ restricted to closed trees. If $f : S \to T$ is map in $\overline{\Omega}$, we define $D(f) : D(S) \to D(T)$ to be the unique root-preserving map $D(S) \to D(T)$ extending $S \to T$, in the sense that the diagram

$$\begin{array}{ccc} D(S) & \xrightarrow{\;D(f)\;} & D(T) \\[2pt] u_S \uparrow & & \uparrow u_T \\[2pt] S & \xrightarrow[\;f\;]{} & T \end{array}$$

commutes. Since this property determines the action of $D(f)$ on the edges of $D(S)$, there is at most one such map. Its existence follows from the fact that for every edge e of $D(T)$ (or more generally of any closed tree), there is a (unique) unary operation from e to the root of $D(T)$ in $\Omega(D(T))$.

One checks that this indeed defines a functor $D : \overline{\Omega} \to \overline{\Omega}$.

Proposition 4.8. *The maps*

$$T \xrightarrow{\;u_T\;} D(T) \xleftarrow{\;\overline{a_T}\;} \overline{\eta}$$

are natural in T in $\overline{\Omega}$.

Proof. The naturality of u_T is true by definition. The one of $\overline{a_T}$ boils down to the fact that for any map f in $\overline{\Omega}$, the map $D(f)$ is root-preserving. $\square$

Remark 4.9. The diagram

$$1_{\overline{\Omega}} \xrightarrow{\;u\;} D \xleftarrow{\;\overline{a}\;} \overline{\eta}$$

is a "split décalage" in the sense of [4, paragraph 3.1]. This implies that $\overline{\Omega}$ is a (strict) test category (see [4, Corollary 3.7]).

Theorem 4.10. *The category Ω is aspherical.*

Proof. By Proposition 4.6, Ω is aspherical if and only if $\overline{\Omega}$ is. But the asphericity of $\overline{\Omega}$ follows from Proposition 4.8. □

5. The category Ω is a test category

In this section, we will prove that Ω is a test category. Our proof is based on the fact that for any tree T, the dendroidal set $\Delta_1 \otimes T$ is aspherical. More generally and parallel to the reasoning in Section 2, we observe that for any two trees S and T, their tensor product $S \otimes T$ is aspherical. To prove this, we will need the shuffle formula introduced in [14, Section 9]:

Proposition 5.1. *Let S and T be two trees. The dendroidal set $S \otimes T$ can be written as a finite union of subdendroidal sets*

$$S \otimes T = \bigcup_{\sigma \in \mathrm{Sh}_{S,T}} F_\sigma$$

satisfying the following properties:
  (a) *the F_σ's are representable;*
  (b) *the F_σ's have the same root and leaves, seen as elements of $(S \otimes T)(\eta)$;*
  (c) *the F_σ's are full subdendroidal sets of $S \otimes T$, where $X \subset Y$ is said to be full if an element of $Y(U)$, for U a tree, belongs to $X(U)$ if and only if all its faces in $Y(\eta)$ belong to $X(\eta)$.*

Remark 5.2. The indexing set $\mathrm{Sh}_{S,T}$ in the above formula is the set of (S,T)-shuffles introduced in [14] under the name of "percolation schemes for S and T".

Proposition 5.3. *If S and T are two trees, then the dendroidal set $S \otimes T$ is aspherical.*

Proof. By Lemma 2.3 applied to the formula

$$S \otimes T = \bigcup_{\sigma \in \mathrm{Sh}_{S,T}} F_\sigma,$$

it suffices to prove that for any non-empty $J \subseteq \mathrm{Sh}_{S,T}$, the dendroidal set

$$F_J = \bigcap_{j \in J} F_j \subset S \otimes T$$

is aspherical. We will actually show that it is representable. As all the F_j's are full subdendroidal sets of $S \otimes T$, the intersection F_J is the (unique) full subdendroidal set such that $F_J(\eta) = \cap_{j \in J} F_j(\eta)$. In particular, for any $j \in J$, F_J is a full subdendroidal set of F_j containing the root and the leaves of F_j. This implies that F_J is an iterated inner face of F_j and is hence representable. □

We will now see that $\Delta_1 \otimes X$ can be thought of as a cylinder object, at least when X is normal.

5.4. If X is a dendroidal set, we have canonical maps

$$X \amalg X \xrightarrow{(\partial^0, \partial^1)} \Delta_1 \otimes X \xrightarrow{\sigma} X,$$

factorising the codiagonal, induced by the diagram

$$\Delta_0 \amalg \Delta_0 = \partial\Delta_1 \hookrightarrow \Delta_1 \to \Delta_0,$$

tensored by X.

Proposition 5.5. *If X is a normal dendroidal set, then*

(a) *the map $(\partial^0, \partial^1) : X \amalg X \to \Delta_1 \otimes X$ is a normal monomorphism;*
(b) *the map $\sigma : \Delta_1 \otimes X \to X$ is a weak equivalence.*

Proof. The first assertion follows since the tensor product of a monomorphism of simplicial sets and a normal dendroidal set is a normal monomorphism (see [9, Section 3.4]).

The second assertion is a special case of Proposition 5.3 if X is representable. Since the functor $X \mapsto \Delta_1 \otimes X$ preserves colimits and monomorphisms between normal objects (see loc. cit.), the general case follows by a standard induction on normal objects (see for instance [3, Proposition 8.2.8]), using the fact that the functor $\lambda_!$ of 1.9 preserves colimits and monomorphisms, and detects weak equivalences. $\qquad\square$

Theorem 5.6. *The category Ω is a test category.*

Proof. Since Ω is aspherical (Theorem 4.10), by Theorem 1.7 it suffices to show that the Lawvere interval L_Ω is locally aspherical; that is, that for every tree T, the projection $p : L_\Omega \times T \to T$ is a weak equivalence. The map p satisfies the two following properties: first, it has the right lifting property with respect to monomorphisms (as L_Ω is injective); second, its source is normal (as its target is). The following standard argument shows that any map $p : X \to Y$ satisfying these two conditions is a weak equivalence. Note first that such a map admits a section s. Consider now the commutative square

$$
\begin{array}{ccc}
X \amalg X & \xrightarrow{\;\;(1, sp)\;\;} & X \\
{\scriptstyle (\partial^0, \partial^1)}\downarrow & & \downarrow{\scriptstyle p} \\
\Delta_1 \otimes X & \xrightarrow[\;\sigma\;]{} X \xrightarrow[\;p\;]{} & Y.
\end{array}
$$

Since by the previous proposition (∂^0, ∂^1) is a monomorphism, this square admits a lifting $h : \Delta_1 \otimes X \to X$ defining a homotopy from 1 to sp in an obvious sense. As $\sigma : \Delta_1 \otimes X \to X$ is a weak equivalence (again by the previous proposition), this implies that sp is a weak equivalence and so p has a left as well as a right inverse in the homotopy category, hence is a weak equivalence, thereby proving the result. $\quad\square$

Remark 5.7. The category Ω is not a *strict* test category (see [11, Section 1.6]) as the category $\Omega/(\eta \times C_2) \simeq \Delta/C_2$, where C_2 is the 2-corolla, is not aspherical (it is not even connected).

Remark 5.8. The planar variation Ω_p of Ω, defined by replacing symmetric operads by non-symmetric operads, is also a test category. Indeed, Ω_p is equivalent to the slice category of Ω over the presheaf of planar structures and by [11, Remark 1.5.4], it suffices to observe that the same argument as in Section 4 shows that Ω_p is aspherical. Similarly, the category $\overline{\Omega_p}$ of closed planar trees is a strict test category.

Corollary 5.9. *There exists a proper combinatorial model category structure on $\widehat{\Omega}$ whose cofibrations are the monomorphisms and whose weak equivalences are the weak equivalences defined in 1.3. Moreover, the functor $\lambda_! : \widehat{\Omega} \to \widehat{\Delta}$ of 1.9 is a left Quillen equivalence.*

Proof. This follows from the previous theorem using Theorems 1.8 and 1.10. □

The model category structure of the previous corollary will be called the *test model category structure*. From now on, its weak equivalences will be called *test weak equivalences* to distinguish them from operadic weak equivalences.

6. Comparison with the operadic model category structure

In this last section, we compare the test model category structure on $\widehat{\Omega}$ with the operadic model category structure.

Proposition 6.1. *Every map of $\widehat{\Omega}$ having the right lifting property with respect to normal monomorphisms is a test weak equivalence.*

Proof. Let $p : X \to Y$ be such a map. By Quillen's Theorem A, to prove that p is a test weak equivalence, it suffices to check that for any tree T and any map $T \to Y$, the map $q = p \times_Y T : X \times_Y T \to T$ is a test weak equivalence. But this map q has the same lifting property as p and, if

$$
\begin{array}{ccc}
A & \longrightarrow & X \times_Y T \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle q} \\
B & \longrightarrow & T
\end{array}
$$

is a commutative square where i is a monomorphism, then i is automatically normal as T and hence B are normal. In particular, q is a trivial fibration in the test model category structure and hence a test weak equivalence, thereby proving the result. □

Theorem 6.2. *There exists a proper model category structure on $\widehat{\Omega}$ whose cofibrations are the normal monomorphisms and whose weak equivalences are the test weak equivalences.*

This model category structure on $\widehat{\Omega}$ will be called the *normal test model category structure*.

Proof. The existence of a model category structure with the announced weak equivalences and cofibrations is a consequence of the following lemma applied to the test model category structure, the hypothesis of the lemma being satisfied by the previous proposition. As properness only depends on the weak equivalences, the properness of the resulting model category structure follows from the properness of the test model category structure. □

Lemma 6.3. *Let $\mathcal{M}$ be a model category. Suppose Cof' is a class of cofibrations, which is the saturation of a set (rather than a class) of morphisms allowing the small object argument, and has the property that any map having the right lifting property with respect to Cof' is a weak equivalence. Then there exists a model category structure on $\mathcal{M}$ with the same weak equivalences, and Cof' as class of cofibrations.*

Proof. This lemma is probably well-known to experts, but we include a proof for completeness. Let us denote by W, Cof and Fib the classes of weak equivalences, cofibrations and fibrations of $\mathcal{M}$, and by Fib' the class of maps having the right lifting property with respect to $\mathsf{Cof}' \cap \mathsf{W}$. Let us show that $(\mathcal{M}, \mathsf{W}, \mathsf{Cof}', \mathsf{Fib}')$ is a

model category. The only axioms that are not obviously true are the lifting axiom and the factorisation axiom.

Let us start by the factorisation axiom. By the small object argument, every map f of $\mathcal{M}$ factors as $f = pi$, where i is in Cof' and p has the right lifting property with respect to Cof'. By hypothesis, such a p is in W. This shows that p is in $\mathsf{Fib}' \cap \mathsf{W}$. As for the second factorisation, if f is map of $\mathcal{M}$, we can write $f = pi$, where i is in $\mathsf{Cof} \cap \mathsf{W}$ and p is in Fib. Using the previous factorisation, we get $i = qj$, where j is in Cof' and q is in $\mathsf{Fib}' \cap \mathsf{W}$. As $\mathsf{Cof}' \subset \mathsf{Cof}$, we have $\mathsf{Fib} \subset \mathsf{Fib}'$, showing that pq is in Fib', so that $f = (pq)j$ is a factorisation of the desired kind.

To conclude the proof, we observe that one half of the lifting axiom holds by definition, while the other half follows by the following standard retract argument. Consider a map f in $\mathsf{Fib}' \cap \mathsf{W}$. We can factor f as pi with i in Cof' and p having the right lifting property with respect to Cof' and hence being in W. By the 2-out-of-3 property, i is in W, and hence in $\mathsf{Cof}' \cap \mathsf{W}$. This implies that f has the right lifting property with respect to i and hence, by the retract lemma, that f is a retract of p, and so that it has the right lifting property with respect to the class Cof'. $\qquad\square$

Proposition 6.4. *Every operadic weak equivalence is a test weak equivalence.*

Proof. Theorem 6.2 implies that the class of test weak equivalences satisfies the first three conditions of Theorem 3.10. This shows that the assertion is equivalent to the fact that Segal core inclusions are test weak equivalences. It thus suffices to prove that for any tree T, the Segal core of T is aspherical. This follows from the fact that the Segal core of T can be constructed by iteratively gluing corollas along η. $\qquad\square$

Theorem 6.5. *The normal test model category structure on $\widehat{\Omega}$ is the left Bousfield localisation of the operadic model category structure by the set of maps between representable dendroidal sets.*

Proof. Let E be a normalisation of the terminal dendroidal set, that is, a normal dendroidal set such that the map p to the terminal dendroidal set has the right lifting property with respect to normal monomorphisms. Consider the adjunction

$$p_! : \widehat{\Omega}/E \rightleftarrows \widehat{\Omega} : p^*,$$

where $p_!$ is the forgetful functor and p^* the functor sending X to $X \times E$. For any dendroidal set X, the projection $X \times E \to X$ has the right lifting property with respect to normal monomorphisms and is hence an operadic weak equivalence. This shows that the unit and the counit of the adjunction $(p_!, p^*)$ are objectwise operadic weak equivalences and we have a Quillen equivalence

$$p_! : \widehat{\Omega}_{\mathrm{oper}}/E \rightleftarrows \widehat{\Omega}_{\mathrm{oper}} : p^*,$$

where $\widehat{\Omega}_{\mathrm{oper}}$ denotes the operadic model category (see also [5, proof of Proposition 3.12]). Note that the fact that E is normal implies that the cofibrations of $\widehat{\Omega}_{\mathrm{oper}}/E$ are the monomorphisms. Consider the left Bousfield localisation of this Quillen equivalence by the set S of maps between representables of $\widehat{\Omega}/E \simeq \widehat{\Omega/E}$. We get a Quillen equivalence

$$p_! : L_S(\widehat{\Omega}_{\mathrm{oper}}/E) \rightleftarrows L_{S'}(\widehat{\Omega}_{\mathrm{oper}}) : p^*,$$

where S' denotes the set of maps between representable dendroidal sets, the unit and counit still being objectwise weak equivalences. As every object of $L_S(\widehat{\Omega}_{\mathrm{oper}}/E)$

is cofibrant, a map of dendroidal sets f is a weak equivalence of $L_{S'}(\widehat{\Omega}_{\mathrm{oper}})$ if and only if $p^*(f)$ is in the class W of weak equivalences of $L_S(\widehat{\Omega}_{\mathrm{oper}}/E)$. Similarly, as $X \times E \to X$ is a test weak equivalence for any dendroidal set X, such a map f is a test weak equivalence if and only if $p^*(f)$ is in the class W_∞ of test weak equivalences of $\widehat{\Omega}/E \simeq \widehat{\Omega/E}$, which is nothing but the class of test weak equivalences of $\widehat{\Omega}$ above E. To conclude the proof, it thus suffices to show the equality $\mathsf{W} = \mathsf{W}_\infty$.

By the previous proposition, the identity functor of $\widehat{\Omega}/E$ is a left Quillen functor from the operadic model category to the normal test model category (or more precisely between their slices). As S belongs to W_∞, the universal property of localisations implies that $\mathsf{W} \subset \mathsf{W}_\infty$.

To prove the converse, we will use the machinery of [3]. Define an Ω/E-*localiser* to be the class of weak equivalences of some combinatorial model category structure on $\widehat{\Omega}/E$ whose cofibrations are the monomorphisms. (By [3, Theorem 1.4.3], this is equivalent to what is called an *accessible* localiser in [3, Section 1.4].) Since E is normal, the category Ω/E is a regular skeletal category in the sense of [3, Definition 8.2.5]. Moreover, as E is aspherical, Ω/E is a test category (see [11, Remark 1.5.4]). In particular, by [3, Proposition 6.4.26 and Proposition 8.2.9], W_∞ is the smallest Ω/E-localiser containing S. As W is a localiser containing S, we have $\mathsf{W}_\infty \subset \mathsf{W}$, thereby ending the proof. □

Remark 6.6. The previous theorem can also be proved as follows. One identifies the operadic model category structure $\widehat{\Omega}_{\mathrm{oper}}$ on dendroidal sets with a localisation of the category of dendroidal spaces (that is, simplicial presheaves on Ω) equipped with the Reedy model category structure. This localisation involves the inner horn inclusions $\Lambda_T^e \hookrightarrow T$ and the map $\{0\} \hookrightarrow J$, where J denotes the nerve of the simply connected groupoid on two objects 0 and 1, see [6, Sections 5 and 6]. Thus the localisation of $\widehat{\Omega}_{\mathrm{oper}}$ by the maps between representables is equivalent to a further localisation of dendroidal spaces by the images $S \to T$ of these maps. One checks that localising the Reedy model category structure on dendroidal spaces by these maps $S \to T$ already makes the inner horn inclusions (or equivalently, the Segal core inclusions) weak equivalences, as well as the image of any Kan–Quillen weak equivalence of simplicial sets under the embedding of simplicial spaces into dendroidal spaces, so in particular $\{0\} \to J$. But the localisation thus obtained describes the homotopy theory of homotopically constant contravariant diagrams of spaces on Ω, hence is equivalent to that of spaces since Ω is aspherical.

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