ON THE HOMOTOPY THEORY OF GROTHENDIECK ∞-GROUPOIDS

DIMITRI ARA

ABSTRACT. We present a slight variation on a notion of weak ∞-groupoid introduced by Grothendieck in *Pursuing Stacks* and we study the homotopy theory of these ∞-groupoids. We prove that the obvious definition for homotopy groups of Grothendieck ∞-groupoids does not depend on any choice. This allows us to give equivalent characterizations of weak equivalences of Grothendieck ∞-groupoids, generalizing a well-known result for strict ∞-groupoids. On the other hand, given a model category $\mathcal{M}$ in which every object is fibrant, we construct, following Grothendieck, a fundamental ∞-groupoid functor Π_∞ from $\mathcal{M}$ to the category of Grothendieck ∞-groupoids. We show that if X is an object of $\mathcal{M}$, then the homotopy groups of $\Pi_\infty(X)$ and of X are canonically isomorphic. We deduce that the functor Π_∞ respects weak equivalences.

INTRODUCTION

The notion of Grothendieck ∞-groupoid has been introduced in 1983 by Grothendieck in a famous letter to Quillen that became the starting point of the equally famous text *Pursuing Stacks* ([11]). For more than twenty years, people thought that Grothendieck's definition was only informal. Maltsiniotis realized in 2006 that this definition is perfectly precise. This original definition is explained in [15]. Maltsiniotis also suggested a simplification of the definition in [16]. All the texts written since then (namely [1] and [17]) also use this simplification. In this article, we use a slight variation on this simplification.

The main motivation of Grothendieck is to generalize to higher dimensions the classical fact that the homotopy 1-type of a topological space is classified by its fundamental groupoid up to equivalence. Given a space X, Grothendieck considers the ∞-graph whose objects are points of X, whose 1-arrows are paths, whose 2-arrows are relative homotopies between paths, whose 3-arrows are relative homotopies between relative homotopies between paths, and so on. This ∞-graph seems to bear an algebraic structure: for instance, one can compose arrows, though in a non-canonical way; these compositions are associative up to non-canonical higher arrows; these higher arrows satisfy higher coherences, again in a non-canonical way; etc. Grothendieck suggests that this ∞-graph should be equipped with the structure of an ∞-groupoid (a notion to be defined) and that this ∞-groupoid, up to equivalence, should classify the homotopy type of X. This is an imprecise statement of Grothendieck's conjecture.

The question now is how to define this structure of ∞-groupoid. There exist definitions of n-groupoids for small n's, obtained by giving explicit generators for coherences. But even for $n = 3$, the standard definition (see [10]) is almost intractable. One has to find another kind of definition.

Here is how Grothendieck proceeds. His idea is to define a category C encoding the algebraic structure of an ∞-groupoid. The category of ∞-groupoids will then be defined as the full subcategory of the category of presheaves on C satisfying some left exactness condition (i.e., some higher Segal condition). This category C will not be unique. This reflects the fact that there is no canonical choice of generators for higher coherences. A category C encoding the theory of ∞-groupoids will be called a coherator. But how to define a coherator?

Grothendieck's main insight is that a very simple principle can be used to generate inductively higher coherences, hence giving a definition for coherators. We will refer to this principle as the "coherences generating principle". Here is how it goes. Suppose G is a (weak) ∞-groupoid, whatever it means. Let X be a globular pasting scheme decorated by arrows of G. For instance, X might be
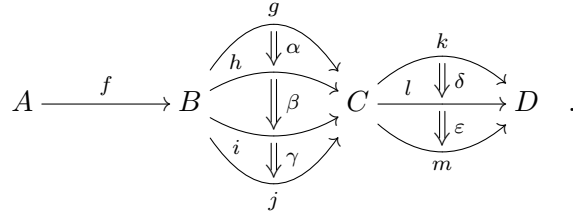
Suppose that from such an X, one can build, using operations of the algebraic structure of ∞-groupoids, two parallel n-arrows Λ and Λ'. Then the coherences generating principle states that there should exist an operation, in the algebraic structure of ∞-groupoid, producing from X an $(n+1)$-arrow going from Λ to Λ'.

For instance, let X be as above and consider

$$\Lambda = \big((\varepsilon \circ \delta) * ((\gamma \circ \beta) \circ \alpha)\big) * 1_f \quad \text{and} \quad \Lambda' = \big((\varepsilon * \gamma) \circ (\delta * (\beta \circ \alpha))\big) * 1_f,$$

where we have denoted by $\circ$ the vertical composition of 2-arrows and by $*$ the horizontal composition of 2-arrows. These two arrows are parallel: their source is $(kg)f$ and their target is $(mj)f$. Hence, the coherences generating principle says that there should exist an operation, in the algebraic structure of ∞-groupoid, producing a 3-arrow

$$\big((\varepsilon \circ \delta) * ((\gamma \circ \beta) \circ \alpha)\big) * 1_f \Rightarrow \big((\varepsilon * \gamma) \circ (\delta * (\beta \circ \alpha))\big) * 1_f$$

from f, α, β, γ, δ, ε fitting in a diagram as above.

If C is a category encoding an algebraic theory satisfying the coherences generating principle, then C describes an algebraic structure where all operations of ∞-groupoids exist, but possibly in a too strict way. A coherator will be defined as a category encoding an algebraic theory freely satisfying the coherences generating principle, that is, a theory obtained by freely adding operations, using the principle, from the operations source and target.

Now that we have given an idea of Grothendieck's definition, let us come back to his original motivation: the classification of homotopy types. Grothendieck shows that if X

is a topological space, then the ∞-graph associated to X can be endowed with the structure of an ∞-groupoid. More precisely, he constructs a fundamental ∞-groupoid functor Π_∞ from the category of topological spaces to the category ∞-$\mathcal{G}$pd of Grothendieck ∞-groupoids. He conjectures that this functor induces an equivalence of categories between the homotopy category of topological spaces Hot and an appropriate localization of ∞-$\mathcal{G}$pd. This conjecture is still not proved. At the end of this article, we give a more precise statement of the conjecture.

A special feature of Grothendieck ∞-groupoids is that they are not defined as ∞-categories satisfying some invertibility conditions. Nevertheless, Maltsiniotis realized that a variation on Grothendieck's definition gives rise to a new notion of weak ∞-category. This notion of ∞-category is closely related to the notion of weak ∞-category introduced by Batanin in [6]. The precise relation between Grothendieck-Maltsiniotis ∞-categories and Batanin ∞-categories is studied in the PhD thesis [1] of the author.

This article is about the homotopy theory of Grothendieck ∞-groupoids. By homotopy theory of ∞-groupoids, we mean the study of the category of ∞-groupoids endowed with weak equivalences of ∞-groupoids. Our contribution to the subject, in addition to foundational aspects, is of three kinds.

First, we propose a slight modification of the definition of Grothendieck ∞-groupoids, whose purpose is to make canonical the inclusion functor from strict ∞-groupoids to Grothendieck ∞-groupoids. This modification takes the form of an additional condition in the definition of an admissible pair. The importance of this modification will be made clear in the forthcoming paper [3].

Second, we prove several foundational results on homotopy groups and weak equivalences of Grothendieck ∞-groupoids. If G is an ∞-groupoid and x is an object of G, the n-th homotopy group $\pi_n(G, x)$ is defined as the group of n-arrows, up to $(n + 1)$-arrows, whose source and target are the iterated unit of x in dimension $n - 1$. This definition depends a priori on several choices. We show that $\pi_n(G, x)$ does not depend on these choices. The heart of the proof is the so-called division lemma. We also deduce from this lemma that a 1-arrow $u : x \to y$ of G induces an isomorphism from $\pi_n(G, x)$ to $\pi_n(G, y)$. Finally, we give four equivalent characterizations of weak equivalences of Grothendieck ∞-groupoids, generalizing a theorem of Simpson on strict n-categories with weak inverses (see Theorem 2.1.III of [20] or Definition 2.2.3 of [21]).

Third, given a model category $\mathcal{M}$ in which every object is fibrant, we construct, following Grothendieck, a fundamental ∞-groupoid functor $\Pi_\infty : \mathcal{M} \to \infty$-$\mathcal{G}$pd, which depends on liftings in $\mathcal{M}$. If X is an object of $\mathcal{M}$, we define a model categorical notion of homotopy groups of X. We show that the homotopy groups of $\Pi_\infty(X)$ are canonically isomorphic to the homotopy groups of X. We deduce that the homotopy groups of $\Pi_\infty(X)$ depend only on X and that the functor Π_∞ respects weak equivalences. In particular, applying this result to the category of topological spaces, we give a precise statement of Grothendieck's conjecture.

Our paper is organized as follows. In the first section, we introduce the globular language, in which the notion of Grothendieck ∞-groupoid will be phrased, and in particular the notion of globular extension. Section 2 is dedicated to the definition of Grothendieck ∞-groupoids. We define in particular contractible globular extensions

and coherators. In Section 3, we explain how to construct structural maps out of a
Grothendieck ∞-groupoid. In particular, we construct enough structural maps to show
that a Grothendieck ∞-groupoid can be truncated to a bigroupoid. In Section 4, we
study the notions of homotopy groups and weak equivalences of ∞-groupoids. We show
that the homotopy groups are well-defined and we give four equivalent characterizations
of weak equivalences. In Section 5, we explain Grothendieck's construction of the funda-
mental ∞-groupoid functor $\Pi_\infty : \mathcal{M} \to \infty\text{-}\mathcal{G}\text{pd}$. We interpret this construction in terms
of Reedy model structures. In Section 6, we recall Quillen's π_1 theory introduced in [19].
We give an alternative formulation in terms of slice categories. In Section 7, we work in
a model category $\mathcal{M}$ in which every object is fibrant. We define a notion of based objects
of $\mathcal{M}$ and, using a loop space construction, we define a theory of homotopy groups for
these based objects. One difference with Quillen's original theory is that our category
$\mathcal{M}$ has no zero object. We show that with our definitions, $\pi_1(X, x)$ is canonically iso-
morphic to $\pi_0(\Omega_x X)$. Finally, in Section 8, we compare the homotopy groups of $\Pi_\infty(X)$
and the homotopy groups of X, where X is an object of a model category in which every
object is fibrant. We finish by a precise statement of Grothendieck's conjecture.

If C is a category, we will denote by C^o the opposite category and by $\widehat{C}$ the category
of presheaves on C. If

$$
\begin{array}{ccccc}
X_1 & & X_2 & \cdots & X_n \\
& \searrow{\scriptstyle f_1} & \nearrow{\scriptstyle g_1} \quad \searrow{\scriptstyle f_2} & & \nearrow{\scriptstyle g_{n-1}} \\
& Y_1 & Y_2 & \cdots \quad Y_{n-1} &
\end{array}
$$

is a diagram in C, we will denote by

$$(X_1, f_1) \times_{Y_1} (g_1, X_2, f_2) \times_{Y_2} \cdots \times_{Y_{n-1}} (g_{n-1}, X_n)$$

its limit. Dually, we will denote by

$$(X_1, f_1) \amalg_{Y_1} (g_1, X_2, f_2) \amalg_{Y_2} \cdots \amalg_{Y_{n-1}} (g_{n-1}, X_n)$$

the colimit of the corresponding diagram in C^o.

1. The globular language

1.1. The globe category. We will denote by $\mathbb{G}$ the *globe category*, that is, the category
generated by the graph

$$
\mathrm{D}_0 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \mathrm{D}_1 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \cdots \underset{\tau_{i-1}}{\overset{\sigma_{i-1}}{\rightrightarrows}} \mathrm{D}_{i-1} \underset{\tau_i}{\overset{\sigma_i}{\rightrightarrows}} \mathrm{D}_i \underset{\tau_{i+1}}{\overset{\sigma_{i+1}}{\rightrightarrows}} \cdots
$$

under the coglobular relations

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i \quad \text{and} \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i, \qquad i \geq 1.$$

For $i \geq j \geq 0$, we will denote by σ_j^i and τ_j^i the morphisms from D_j to D_i defined by

$$\sigma_j^i = \sigma_i \cdots \sigma_{j+2}\sigma_{j+1} \quad \text{and} \quad \tau_j^i = \tau_i \cdots \tau_{j+2}\tau_{j+1}.$$

1.2. Globular sets. The category of *globular sets* or ∞-*graphs* is the category $\widehat{\mathbb{G}}$ of presheaves on $\mathbb{G}$. The data of a globular set X amounts to the data of a diagram of sets

$$\cdots \underset{t_{i+1}}{\overset{s_{i+1}}{\rightrightarrows}} X_i \underset{t_i}{\overset{s_i}{\rightrightarrows}} X_{i-1} \underset{t_{i-1}}{\overset{s_{i-1}}{\rightrightarrows}} \cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} X_1 \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_0$$

satisfying the globular relations

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}, \qquad i \geq 1.$$

For $i \geq j \geq 0$, we will denote by s_j^i and t_j^i the maps from X_i to X_j defined by

$$s_j^i = s_{j+1} \cdots s_{i-1} s_i \quad \text{and} \quad t_j^i = t_{j+1} \cdots t_{i-1} t_i.$$

If X is a globular set, we will call X_0 the set of *objects* of X and, for $i \geq 0$, X_i the set of *i-arrows*. If u is an i-arrow of X for $i \geq 1$, $s_i(u)$ (resp. $t_i(u)$) will be called the *source* (resp. the *target*) of u. We will often denote an arrow u of X whose source is x and whose target is y by $u : x \to y$.

1.3. Globular sums. Let n be a positive integer. A *table of dimensions* of width n is the data of integers $i_1, \ldots, i_n, i_1', \ldots, i_{n-1}'$ such that

$$i_k > i_k' \quad \text{and} \quad i_{k+1} > i_k', \qquad 1 \leq k \leq n-1.$$

We will denote such a table of dimensions by

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_n \\ & i_1' & & i_2' & \cdots & i_{n-1}' & \end{pmatrix}.$$

The *dimension* of such a table is the greatest integer appearing in the table.

Let (C, F) be a category under $\mathbb{G}$, that is, a category C endowed with a functor $F : \mathbb{G} \to C$. We will denote in the same way the objects and morphisms of $\mathbb{G}$ and their image by the functor F. Let

$$T = \begin{pmatrix} i_1 & & i_2 & & \cdots & & i_n \\ & i_1' & & i_2' & \cdots & i_{n-1}' & \end{pmatrix}$$

be a table of dimensions. The *globular sum* in C associated to T (if it exists) is the iterated amalgamated sum

$$(\mathrm{D}_{i_1}, \sigma_{i_1'}^{i_1}) \amalg_{\mathrm{D}_{i_1'}} (\tau_{i_1'}^{i_2}, \mathrm{D}_{i_2}, \sigma_{i_2'}^{i_2}) \amalg_{\mathrm{D}_{i_2'}} \ldots \amalg_{\mathrm{D}_{i_{n-1}'}} (\tau_{i_{n-1}'}^{i_n}, \mathrm{D}_{i_n})$$

in C, that is, the colimit of the diagram

in C. We will denote it briefly by

$$\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i_1'}} \mathrm{D}_{i_2} \amalg_{\mathrm{D}_{i_2'}} \ldots \amalg_{\mathrm{D}_{i_{n-1}'}} \mathrm{D}_{i_n}.$$

1.4. Globular extensions. A category C under $\mathbb{G}$ is said to be a *globular extension* if for every table of dimensions T (of any width), the globular sum associated to T exists in C.

Let C and D be two globular extensions. A *morphism of globular extensions* from C to D is a functor from C to D under $\mathbb{G}$ (that is, such that the triangle

$$
\begin{array}{ccc}
 & \mathbb{G} & \\
\swarrow & & \searrow \\
C & \longrightarrow & D
\end{array}
$$

commutes) which respects globular sums. We will also call such a functor a *globular functor*.

Examples 1.5.

1. If C is a cocomplete category and $F : \mathbb{G} \to C$ is any functor, then (C, F) is a globular extension.

2. Let $\mathcal{T}op$ be the category of topological spaces. We define a functor $R : \mathbb{G} \to \mathcal{T}op$ in the following way. For $i \geq 0$, the object D_i is sent by R to the i-dimensional ball

$$\mathbf{D}_i = \{x \in \mathbb{R}^i; \|x\| \leq 1\}.$$

For $i \geq 1$, the morphisms σ_i and τ_i are sent by R respectively to $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ defined by

$$\boldsymbol{\sigma}_i(x) = (x, \sqrt{1 - \|x\|^2}) \quad \text{and} \quad \boldsymbol{\tau}_i(x) = (x, -\sqrt{1 - \|x\|^2}), \qquad x \in \mathbf{D}_{i-1}.$$

These morphisms are the inclusions of the two hemispheres of $\mathbf{D}_i$. One instantly checks that these maps satisfy the coglobular relations and our functor R is thus well-defined. Note that this functor is faithful. The category $\mathcal{T}op$ being cocomplete, $(\mathcal{T}op, R)$ is a globular extension. In what follows, $\mathcal{T}op$ will always be endowed with this globular extension structure.

3. Let $h : \mathbb{G} \to \widehat{\mathbb{G}}$ be the Yoneda functor. Then $(\widehat{\mathbb{G}}, h)$ is a globular extension. The globular sums for this globular extension are the globular sets T^* (where T is a finite planar rooted tree) introduced by Batanin in [6].

4. Let Θ_0 be the full category of $\widehat{\mathbb{G}}$ whose objects consist of a choice of a globular sum (which is only defined up to isomorphism) for every table of dimensions. The category Θ_0 is obviously endowed with the structure of a globular extension. This category is canonically isomorphic to the category Θ_0 defined in terms of finite planar rooted trees by Berger in [7].

We will see that the globular extension Θ_0 is the initial globular extension in some 2-categorical sense (Proposition 1.6). See also Proposition 3.2 of [2] for a more abstract point of view on Θ_0.

5. Let Θ be the full subcategory of the category of strict ∞-categories whose objects are free strict ∞-categories on objects of Θ_0. The category Θ is canonically endowed with the structure of a globular extension. This category is canonically isomorphic to the cell category introduced by Joyal in [13], as was proved independently by Makkai and Zawadowski in [14] and by Berger in [7]. Alternative definitions of Θ are given in [7] and [8]. See also Proposition 3.11 of [2] for a definition of Θ by universal property.

6. Let $\widetilde{\Theta}$ be the full subcategory of the category of strict ∞-groupoids whose objects are free strict ∞-groupoids on objects of Θ_0. The category $\widetilde{\Theta}$ is canonically endowed

with the structure of a globular extension. The category $\widetilde{\Theta}$ can be thought of as a groupoidal analogue to Joyal's cell category. See Proposition 3.18 of [2] for a definition of $\widetilde{\Theta}$ by universal property.

Proposition 1.6. *The globular extension Θ_0 has the following universal property: for every globular extension (C, F), there exists a globular functor $F_0 : \Theta_0 \to C$, unique up to a unique natural transformation, such that the triangle*

$$
\begin{array}{ccc}
\Theta_0 & & \\
\uparrow & \searrow^{F_0} & \\
\mathbb{G} & \xrightarrow{\ F\ } & C
\end{array}
$$

commutes.

Proof. See Proposition 3.2 and the next paragraph of [2]. $\qquad\square$

Remark 1.7. If (C, F) is a globular extension, a lifting $F_0 : \Theta_0 \to C$, as in the above universal property, amounts to the choice of a globular sum in C for every table of dimensions.

1.8. Globular presheaves. Let C be a globular extension. A *globular presheaf* on C or *model* of C is a presheaf $X : C^{\mathrm{o}} \to \mathcal{S}et$ on C such that the functor $X^{\mathrm{o}} : C \to \mathcal{S}et^{\mathrm{o}}$ respects globular sums, i.e., such that for every table of dimensions

$$
T = \begin{pmatrix} i_1 & & i_2 & & \cdots & & i_n \\ & i'_1 & & i'_2 & \cdots & i'_{n-1} & \end{pmatrix},
$$

the canonical map

$$
X(\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \ldots \amalg_{\mathrm{D}_{i'_{n-1}}} \mathrm{D}_{i_n}) \to X_{i_1} \times_{X_{i'_1}} \cdots \times_{X_{i'_{n-1}}} X_{i_n}
$$

is a bijection. We will denote by $\mathrm{Mod}(C)$ the full subcategory of the category $\widehat{C}$ of presheaves on C whose objects are globular presheaves.

The canonical functor $\mathbb{G} \to C$ induces a functor $\widehat{C} \to \widehat{\mathbb{G}}$ which restricts to a functor $\mathrm{Mod}(C) \to \widehat{\mathbb{G}}$. If X is a globular presheaf on C, the image of X by this functor will be called the *underlying globular set* of X. We will often implicitly apply the underlying globular set functor to transfer notation and terminology from globular sets to globular presheaves. For instance, we will denote $X(\mathrm{D}_i)$ by X_i and we will call this set the set of i-arrows of X.

Examples 1.9.

1. The category of globular presheaves on Θ_0 is canonically equivalent to the category of globular sets. More precisely, the composition

$$
\mathrm{Mod}(\Theta_0) \to \widehat{\Theta_0} \xrightarrow{i^*} \widehat{\mathbb{G}},
$$

where i^* denotes the restriction functor induced by the canonical functor $i : \mathbb{G} \to \Theta_0$, is an equivalence of categories. See Lemma 1.6 of [7] or Proposition 3.5 of [2].

2. The category of globular presheaves on Θ is canonically equivalent to the category of strict ∞-categories. See Theorem 1.12 of [7] or Proposition 3.14 of [2].

3. The category of globular presheaves on $\widetilde{\Theta}$ is canonically equivalent to the category of strict ∞-groupoids. See Proposition 3.21 of [2].

1.10. Globular extensions under Θ_0**.** A *globular extension under* Θ_0 is a category C endowed with a functor $\Theta_0 \to C$ such that $(C, \mathbb{G} \to \Theta_0 \to C)$ is a globular extension. If C is a globular extension under Θ_0, the globular sum associated to a table of dimensions is uniquely defined. A *morphism of globular extensions under* Θ_0 is a functor under Θ_0 between globular extensions under Θ_0. Such a functor automatically respects globular sums.

Proposition 1.11. *Let* C *be a category under* Θ_0*. There exists a globular extension* $\overline{C}$ *under* Θ_0*, endowed with a functor* $C \to \overline{C}$ *under* Θ_0 *having the following universal property: for every globular extension* D *under* Θ_0*, endowed with a functor* $C \to D$ *under* Θ_0*, there exists a unique functor* $\overline{C} \to D$ *such that the triangle*

$$
\begin{array}{ccc}
 & & \overline{C} \\
 & \nearrow & \downarrow \\
C & \searrow & \\
 & & D
\end{array}
$$

commutes.

Proof. This is a special case of a standard categorical construction (see Proposition 3 of [5]). See also Section 2.6 of [1] and paragraph 3.10 of [17] for this particular case. $\square$

1.12. Globular completion. If C is a category under Θ_0, the globular extension $\overline{C}$ of the previous proposition (which is unique up to a unique isomorphism) will be called the *globular completion* of C. Note that the functor $C \to \overline{C}$ is bijective on objects.

Remark 1.13. The theory of globular extensions is in some sense generated by the category $\mathbb{G}$ and the diagrams in $\mathbb{G}$ describing globular sums. More precisely, starting from a category I and a set D of small diagrams in I, there is an obvious generalization of the theory of globular extensions to a theory of (I, D)-extensions. When I is the terminal category and D is the set of diagrams describing finite sums, we obtain (up to a variance issue) the theory of Lawvere theories. In this general setting, the category Θ_0 (resp. the globular extensions C under Θ_0 such that $\Theta_0 \to C$ is bijective on objects) plays the same role as a skeleton of the category of finite sets (resp. as Lawvere theories).

2. GROTHENDIECK ∞-GROUPOIDS

2.1. Globularly parallel arrows and liftings. Let C be a globular extension. If $f : \mathrm{D}_n \to X$ is a morphism of C whose source is a D_n, $n \geq 1$, then the *globular source* (resp. the *globular target*) of f is the morphism $f\sigma_n : \mathrm{D}_{n-1} \to X$ (resp. $f\tau_n : \mathrm{D}_{n-1} \to X$).

If $f, g : \mathrm{D}_n \to X$ are two morphisms of C whose source is a D_n, $n \geq 0$, we will say that f and g are *globularly parallel* if, either $n = 0$, or $n \geq 1$ and f, g have the same globular source and globular target.

Let now $(f, g) : \mathrm{D}_n \to X$ be a pair of morphisms of C. A *lifting* of the pair (f, g) is a morphism $h : \mathrm{D}_{n+1} \to X$ whose globular source is f and whose globular target is g,

that is, such that the inner and the outer triangles of the diagram

$$\begin{array}{ccc} \mathrm{D}_{n+1} & & \\ {\scriptstyle\tau_{n+1}}\Big\uparrow\Big\uparrow{\scriptstyle\sigma_{n+1}} & \searrow{\scriptstyle h} & \\ \mathrm{D}_n & \underset{g}{\overset{f}{\rightrightarrows}} & X \end{array}$$

commute. The existence of such a lifting obviously implies that f and g are globularly parallel.

Example 2.2. Let $C = \mathcal{T}op$. Two maps $f, g : \mathbf{D}_n \to X$ are globularly parallel if their restrictions to the boundary $\partial \mathbf{D}_n$ of $\mathbf{D}_n$ coincide, i.e., if they induce a map

$$(f, g) : \partial \mathbf{D}_{n+1} = \mathbf{D}_n \amalg_{\partial \mathbf{D}_n} \mathbf{D}_n \to X.$$

A lifting of the pair (f, g) corresponds to a lifting of the induced map $\partial \mathbf{D}_{n+1} \to X$ to $\mathbf{D}_{n+1}$, that is, to a map $h : \mathbf{D}_{n+1} \to X$ such that the triangle

$$\begin{array}{ccc} \mathbf{D}_{n+1} & & \\ \uparrow & \searrow{\scriptstyle h} & \\ \partial \mathbf{D}_{n+1} & \underset{(f,g)}{\longrightarrow} & X \end{array}$$

commutes. Note that when X is fixed, such an h exists for every $n \geq 0$ and every $(f, g) : \mathbf{D}_n \to X$ globularly parallel if and only if X is weakly contractible.

2.3. Admissible pairs. Let C be a globular extension. A pair of morphisms

$$(f, g) : \mathrm{D}_n \to S$$

of C whose source is a D_n, $n \geq 0$, is said to be $(\infty, 0)$-*admissible*, or briefly *admissible*, if

- the morphisms f and g are globularly parallel;
- the object S is a globular sum;
- the dimension of S is less than or equal to $n + 1$.

An admissible pair is *strictly admissible* if it does not admit a lifting.

2.4. Contractible globular extensions. We will say that a globular extension C is $(\infty, 0)$-*contractible*, or briefly *contractible*, if every admissible pair of C admits a lifting. Such a globular extension is called a pseudo-coherator in [17] and [1].

2.5. ∞-groupoids of type C. Let C be a contractible globular extension. The category of ∞-*groupoids of type C* is the category $\mathrm{Mod}(C)$ of globular presheaves on C. We will denote it in a more suggestive way by ∞-$\mathcal{G}\mathrm{pd}_C$.

Remark 2.6. It might be more reasonable to define ∞-groupoids of type C only when objects of C are in bijection with tables of dimensions. We chose not to do so for technical reasons (see for instance Proposition 5.5).

Examples 2.7.
1. The globular extension $\mathcal{T}op$ is contractible. Indeed, it is obvious that every globular sum in $\mathcal{T}op$ is contractible in the topological sense (see Proposition 5.11 for a proof in a

more general setting). The contractibility of $\mathcal{T}op$ then follows from the last assertion of Example 2.2.

2. We will prove in [3] that the globular extension $\widetilde{\Theta}$ is contractible. More precisely, we will show that every admissible pair in $\widetilde{\Theta}$ admits a *unique* lifting. This will allow us to define a canonical inclusion functor of strict ∞-groupoids into ∞-groupoids of type C, where C is a coherator endowed with a defining tower (see Remark 2.15).

3. The globular extension Θ is *not* $(\infty,0)$-contractible. Indeed, the admissible pair (τ_n, σ_n), $n \geq 1$, does not admit a lifting. This reflects the fact that globular presheaves on Θ, i.e., strict ∞-categories, do not have inverses. Nevertheless, the globular extension Θ is (∞,∞)-contractible in some sense (admissible for a theory of ∞-categories in the terminology of [17] and [1]). See Proposition 5.1.5 of [1].

Remarks 2.8.

1. The definition of admissible pairs differs from the one given in the previous texts on Grothendieck ∞-groupoids: a dimensional condition has been added. The purpose of this condition is to make unique the lifting of an admissible pair in $\widetilde{\Theta}$ and hence to make canonical the inclusion functor of strict ∞-groupoids into Grothendieck ∞-groupoids.

2. The category ∞-$\mathcal{G}\mathrm{pd}_C$ should *not* be thought of as a category of *weak* ∞-groupoids unless C satisfies some freeness condition. Indeed, for $C = \widetilde{\Theta}$, the category ∞-$\mathcal{G}\mathrm{pd}_C$ is nothing but the category of strict ∞-groupoids. One way to define weak ∞-groupoids without defining this freeness condition would be to define the category of weak ∞-groupoids as the "union" of all the ∞-$\mathcal{G}\mathrm{pd}_C$'s, where C varies among contractible globular extensions.

2.9. Adding liftings to globular extensions. Let C be a globular extension and let A be a set of admissible pairs of C. We will denote by $C[h_A]^\flat$ the category obtained from C by formally adding a lifting $h_{(f,g)}$ to every pair (f,g) in A. More precisely, the category $C[h_A]^\flat$ is the category, endowed with a functor $C \to C[h_A]^\flat$ such that the image of every pair of A admits a lifting in $C[h_A]^\flat$, satisfying the following universal property: for every category D, endowed with a functor $C \to D$ such that the image of every pair of A admits a lifting in D, there exists a unique functor $C[h_A]^\flat \to D$ such that the triangle

$$
\begin{array}{ccc}
 & & C[h_A]^\flat \\
 & \nearrow & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

commutes.

The category $C[h_A]^\flat$ is naturally a category under $\mathbb{G}$ but it has no reason to be a globular extension. Let us assume that the globular extension C is a globular extension under Θ_0. Then $C[h_A]^\flat$ is also a category under Θ_0 and we can consider its globular completion $\overline{C[h_A]^\flat}$. We will denote this globular extension under Θ_0 by $C[h_A]$. The category $C[h_A]$ has the following universal property: for every globular extension D over Θ_0, endowed with a functor $C \to D$ under Θ_0 such that the image of every admissible pair of

A admits a lifting in D, there exists a unique functor $C[h_A] \to D$ such that the triangle

commutes. Note that the functor $C \to C[h_A]$ is bijective on objects.

2.10. Free globular extensions. A *cellular tower* of globular extensions is a tower of globular extensions

$$C_0 = \Theta_0 \to C_1 \to \cdots \to C_n \to \ldots \quad ,$$

endowed, for each $n \geq 0$, with a set A_n of admissible pairs of C_n such that

$$C_{n+1} = C_n[h_{A_n}].$$

Such a tower is entirely defined by the A_n's. We will say that a cellular tower (C_n, A_n) *defines a globular extension* C if C is isomorphic to $\varinjlim C_n$. In this case, we will also say that (C_*, A_*) is a *defining tower* of C.

We will say that a globular extension C is *free* if C admits a defining tower. Note that if C is free, a defining tower of C gives a functor $\Theta_0 \to C$ which is bijective on objects.

2.11. Coherators. An $(\infty, 0)$-*coherator*, or briefly a *coherator*, is a globular extension which is free and contractible.

Examples 2.12. Let (C_*, A_*) be a cellular tower. If the A_n's are such that every admissible pair of $C = \varinjlim C_n$ comes from an A_n, then C is a coherator. This remark allows us to define three coherators:

- The canonical coherator: A_n is the set of all admissible pairs of C_n.
- The reduced canonical coherator: A_n is the set of all strictly admissible pairs of C_n.
- The Batanin-Leinster coherator: A_n is the set of the admissible pairs that do not come from an A_m, $m \leq n$, via the functor $C_m \to C_n$. The name of this coherator comes from the relation it bears with Batanin-Leinster ∞-categories (see Section 6.7 of [1]).

Remark 2.13. If C is a coherator, the category of ∞-groupoids of type C can be thought of as a category of *weak* ∞-groupoids. The fact that this category depends on a coherator reflects the non-uniqueness of the choice of generators for higher coherences. Nevertheless, if C and C' are two coherators, the category ∞-$\mathcal{G}\text{pd}_C$ and ∞-$\mathcal{G}\text{pd}_{C'}$ should be equivalent in some weak sense to be defined. Note that Grothendieck's conjecture (Conjecture 8.14) implies that their homotopy categories are equivalent.

Proposition 2.14. *Let C be a free globular extension. For any contractible globular extension D, there exists a globular functor $C \to D$.*

Proof. Let (C_*, A_*) be a cellular tower defining C. By the universal property of Θ_0, the functor $\mathbb{G} \to D$ lifts to a globular functor $F_0 : C_0 = \Theta_0 \to D$. Suppose now by induction that we have a globular functor $F_n : C_n \to D$. Every admissible pair of A_n is sent to an admissible pair of D. Since D is contractible, every such pair admits a lifting, and, by the universal property of $C_{n+1} = C_n[h_{A_n}]$, we can lift F_n to a globular functor

$F_{n+1} : C_{n+1} \to D$. We hence get a functor $F = \varinjlim F_n : C \to D$ which is obviously globular. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remarks 2.15.

1. The globular functor $C \to D$ of the previous proposition is not unique: it depends on a cellular tower defining C and on a choice of liftings. Nevertheless, if such a choice is made, the globular functor $C \to D$ becomes unique (up to a unique isomorphism).

2. In particular, if C is a coherator endowed with a defining tower and $D = \widetilde{\Theta}$, by Example 2.7.2, there exists a unique functor $F : C \to \widetilde{\Theta}$ under Θ_0. This functor induces a functor from strict ∞-groupoids to ∞-groupoids of type C. This is the canonical inclusion functor of strict ∞-groupoids into ∞-groupoids of type C.

3. Some structural maps of Grothendieck ∞-groupoids

3.1. In this section, we fix a contractible globular extension C and an ∞-groupoid G of type C. The purpose of the section is to convince the reader that G deserves to be called an ∞-groupoid. For this purpose, we will explain how to construct structural maps (i.e., operations and coherences) for G out of C. More precisely, we will show that G can be endowed with compositions, units and inverses, and that these operations satisfy the axioms of strict ∞-groupoids up to coherences. We will also give examples of higher coherences between these coherences.

3.2. First example: codimension 1 compositions. Let $i \geq 1$. We will explain how to endow G with a composition of i-arrows in codimension 1, i.e., with a map

$$*_{i-1}^{i} : G_i \times_{G_{i-1}} G_i \to G_i$$

sending i-arrows

$$x \xrightarrow{u} y \xrightarrow{v} z$$

to an i-arrow

$$v *_{i-1}^{i} u : x \to z.$$

Let $p_1, p_2 : G_i \times_{G_{i-1}} G_i \to G_i$ denote the canonical projections. The conditions on the source and the target of $*_{i-1}^{i}$ can be rewritten in the following way:

$$s_i *_{i-1}^{i} = s_i p_2 \quad \text{and} \quad t_i *_{i-1}^{i} = t_i p_1.$$

The important fact that will allow us to construct our map $*_{i+1}^{i}$ is that the maps $s_i p_2$ and $t_i p_1$ are induced by morphisms of C. Indeed, denote by $\varepsilon_1, \varepsilon_2 : D_i \to D_i \amalg_{D_{i-1}} D_i$ the canonical morphisms. Since G is a globular presheaf, the canonical morphism

$$j : G(D_i \amalg_{D_{i-1}} D_i) \to G_i \times_{G_{i-1}} G_i$$

is a bijection and we have

$$G(\varepsilon_i) = p_i j, \quad i = 1, 2.$$

It follows that

$$G(\varepsilon_2 \sigma_i) j^{-1} = s_i p_2 \quad \text{and} \quad G(\varepsilon_1 \tau_i) j^{-1} = t_i p_1.$$

Consider now the pair

$$(\varepsilon_2 \sigma_i, \varepsilon_1 \tau_i) : D_{i-1} \to D_i \amalg_{D_{i-1}} D_i$$

of morphisms of C. We claim that this pair is admissible. For $i = 1$, there is nothing to check. For $i \geq 2$, we have

$$\varepsilon_2 \sigma_i \sigma_{i-1} = \varepsilon_2 \tau_i \sigma_{i-1} = \varepsilon_1 \sigma_i \sigma_{i-1} = \varepsilon_1 \tau_i \sigma_{i-1}$$

and

$$\varepsilon_2 \sigma_i \tau_{i-1} = \varepsilon_2 \tau_i \tau_{i-1} = \varepsilon_1 \sigma_i \tau_{i-1} = \varepsilon_1 \tau_i \tau_{i-1}.$$

Since C is contractible, this pair admits a lifting in C. It follows that there exists a morphism

$$\nabla_{i-1}^i : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$$

in C such that

$$\nabla_{i-1}^i \sigma_i = \varepsilon_2 \sigma_i \quad \text{and} \quad \nabla_{i-1}^i \tau_i = \varepsilon_1 \tau_i.$$

This morphism will also be denoted by ∇_i. It induces a map

$$*_{i-1}^i : G_i \times_{G_{i-1}} G_i \xrightarrow{j^{-1}} G(\mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i) \xrightarrow{G(\nabla_i)} G_i$$

which has the desired source and target. Indeed, we have

$$s_i *_{i-1}^i = G(\sigma_i) G(\nabla_i) j^{-1} = G(\nabla_i \sigma_i) j^{-1} = G(\varepsilon_2 \sigma_i) j^{-1} = s_i p_2$$

and

$$t_i *_{i-1}^i = G(\tau_i) G(\nabla_i) j^{-1} = G(\nabla_i \tau_i) j^{-1} = G(\varepsilon_1 \tau_i) j^{-1} = t_i p_1.$$

Note that the composition $*_{i-1}^i$ depends on the choice of the lifting ∇_i. Nevertheless, it is easy to show that this composition is unique up to $(i+1)$-arrows. See Proposition 4.6 for details.

3.3. The general pattern. Let $i \geq 1$ and let

$$\begin{pmatrix} i_1 & i_2 & \cdots & & i_n \\ & i_1' & i_2' & \cdots & i_{n-1}' & \end{pmatrix}$$

be a table of dimensions. Suppose we want to construct a structural map

$$m : G_{i_1} \times_{G_{i_1'}} \cdots \times_{G_{i_{n-1}'}} G_{i_n} \to G_i$$

such that

$$s_i m = f \quad \text{and} \quad t_i m = g,$$

where

$$f, g : G_{i_1} \times_{G_{i_1'}} \cdots \times_{G_{i_{n-1}'}} G_{i_n} \to G_{i-1}$$

are two fixed maps. To do so, we first have to find morphisms

$$\varphi, \gamma : \mathrm{D}_{i-1} \to \mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i_1'}} \ldots \amalg_{\mathrm{D}_{i_{n-1}'}} \mathrm{D}_{i_n}$$

in C such that

$$G(\varphi) j^{-1} = f \quad \text{and} \quad G(\gamma) j^{-1} = g,$$

where

$$j : G(\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i_1'}} \ldots \amalg_{\mathrm{D}_{i_{n-1}'}} \mathrm{D}_{i_n}) \to G_{i_1} \times_{G_{i_1'}} \cdots \times_{G_{i_{n-1}'}} G_{i_n}$$

denote the canonical morphism. We must then check that the pair (φ, γ) is admissible (this will be the case if the structural map m is "reasonable"). Then, any lifting

$$\mu : \mathrm{D}_i \to \mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i_1'}} \ldots \amalg_{\mathrm{D}_{i_{n-1}'}} \mathrm{D}_{i_n}$$

in C of the pair (ϕ, γ) will induce a map

$$m : G_{i_1} \times_{G_{i'_1}} \cdots \times_{G_{i'_{n-1}}} G_{i_n} \xrightarrow{j^{-1}} G(\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \ldots \amalg_{\mathrm{D}_{i'_{n-1}}} \mathrm{D}_{i_n}) \xrightarrow{G(\mu)} G_i$$

with the desired source and target.

3.4. In the rest of this section, we will assume that our contractible globular extension C is the canonical coherator. The canonical cellular tower defining C will be denoted by (C_*). It should be clear to the reader that all the structural morphisms that we will define in C exist in every contractible globular extension. Our exposition only uses the canonical coherator to highlight the natural hierarchy between structural maps. For instance, since the pair

$$(\varepsilon_2 \sigma_i, \varepsilon_1 \tau_i) : \mathrm{D}_{i-1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$$

considered in paragraph 3.2 actually comes from $C_0 = \Theta_0$, it admits a lifting in C_1. The composition $*_{i-1}^i$ is hence in some sense a primary operation.

3.5. Examples of structural maps appearing in C_1.
- *Codimension* 1 *compositions*
See paragraph 3.2.
- *Units*
Let $i \geq 0$. The pair

$$(1_{\mathrm{D}_i}, 1_{\mathrm{D}_i}) : \mathrm{D}_i \to \mathrm{D}_i$$

of morphisms of C_0 is obviously admissible. Hence there exists a lifting

$$\kappa_i : \mathrm{D}_{i+1} \to \mathrm{D}_i$$

in C_1 such that

$$\kappa_i \sigma_{i+1} = 1_{\mathrm{D}_i} \quad \text{and} \quad \kappa_i \tau_{i+1} = 1_{\mathrm{D}_i}.$$

This morphism induces a structural map

$$k_i : G_i \to G_{i+1}$$

of G for units. This map sends an i-arrow u to an $(i+1)$-arrow

$$k_i(u) : u \to u.$$

We will see that $k_i(u)$ is a unit for the composition $*_i^{i+1}$ up to an $(i+2)$-arrow.
- *Codimension* 1 *inverses*
Let $i \geq 1$. It follows from the coglobular identities that the pair

$$(\tau_i, \sigma_i) : \mathrm{D}_{i-1} \to \mathrm{D}_i$$

of morphisms of C_0 is admissible. Hence there exists a lifting

$$\Omega_{i-1}^i : \mathrm{D}_i \to \mathrm{D}_i$$

in C_1 such that

$$\Omega_{i-1}^i \sigma_i = \tau_i \quad \text{and} \quad \Omega_{i-1}^i \tau_i = \sigma_i.$$

This morphism will also be denoted by Ω_i. It induces a structural map

$$w_{i-1}^i : G_i \to G_i$$

of G for codimension 1 inverses. This map sends an i-arrow $u : x \to y$ to an i-arrow

$$w_{i-1}^i(u) : y \to x.$$

We will see that $w_{i-1}^i(u)$ is an inverse of u for the composition $*_{i-1}^i$ up to an $(i+1)$-arrow.

Remark 3.6. The list of structural maps appearing in C_1 we have given above is not exhaustive: n-ary compositions or operations mixing compositions, units and inverses are other examples.

3.7. Examples of structural maps appearing in C_2.

- *Codimension 2 compositions*

Let $i \geq 2$. The pair

$$((\varepsilon_1 \sigma_i, \varepsilon_2 \sigma_i) \nabla_{i-1}, (\varepsilon_1 \tau_i, \varepsilon_2 \tau_i) \nabla_{i-1}) : D_{i-1} \to D_i \amalg_{D_{i-2}} D_i$$

of morphisms of C_1, where $\varepsilon_1, \varepsilon_2 : D_i \to D_i \amalg_{D_{i-2}} D_i$ denote the canonical morphisms, is admissible. Indeed, we have

$$
\begin{aligned}
(\varepsilon_1 \sigma_i, \varepsilon_2 \sigma_i) \nabla_{i-1} \sigma_{i-1} &= (\varepsilon_1 \sigma_i, \varepsilon_2 \sigma_i) \varepsilon_2 \sigma_{i-1} \\
&= \varepsilon_2 \sigma_i \sigma_{i-1} = \varepsilon_2 \tau_i \sigma_{i-1} \\
&= (\varepsilon_1 \tau_i, \varepsilon_2 \tau_i) \varepsilon_2 \sigma_{i-1} \\
&= (\varepsilon_1 \tau_i, \varepsilon_2 \tau_i) \nabla_{i-1} \sigma_{i-1}.
\end{aligned}
$$

In the same way, we get

$$(\varepsilon_1 \sigma_i, \varepsilon_2 \sigma_i) \nabla_{i-1} \tau_{i-1} = \varepsilon_1 \tau_i \tau_{i-1} = (\varepsilon_1 \tau_i, \varepsilon_2 \tau_i) \nabla_{i-1} \tau_{i-1}.$$

Hence there exists a lifting

$$\nabla_{i-2}^i : D_i \to D_i \amalg_{D_{i-2}} D_i$$

in C_2 such that

$$\nabla_{i-2}^i \sigma_i = (\varepsilon_1 \sigma_i, \varepsilon_2 \sigma_i) \nabla_{i-1} \quad \text{and} \quad \nabla_{i-2}^i \tau_i = (\varepsilon_1 \tau_i, \varepsilon_2 \tau_i) \nabla_{i-1}.$$

This morphism induces a structural map

$$*_{i-2}^i : G_i \times_{G_{i-2}} G_i \to G_i$$

of G for codimension 2 composition. This map sends i-arrows

$$x \underset{v}{\overset{u}{\Longrightarrow}}_{\Downarrow \alpha} y \underset{v'}{\overset{u'}{\Longrightarrow}}_{\Downarrow \alpha'} z$$

to an i-arrow

$$\alpha' *_{i-2}^i \alpha : y \underset{v' *_{i-2}^{i-1} v}{\overset{u' *_{i-2}^{i-1} u}{\Longrightarrow}}_{\Downarrow} x \quad .$$

- *Codimension 2 inverses*

Let $i \geq 2$. The pair

$$(\sigma_i \Omega_{i-1}, \tau_i \Omega_{i-1}) : D_{i-1} \to D_i$$

of morphisms of C_1 is admissible. Indeed, we have

$$\sigma_i \Omega_{i-1} \sigma_{i-1} = \sigma_i \tau_{i-1} = \tau_i \tau_{i-1} = \tau_i \Omega_{i-1} \sigma_{i-1}.$$

In the same way, we get

$$\sigma_i \Omega_{i-1} \tau_{i-1} = \sigma_i \sigma_{i-1} = \tau_i \Omega_{i-1} \tau_{i-1}.$$

Hence there exists a lifting

$$\Omega_{i-2}^i : D_i \to D_i$$

in C_2 such that

$$\Omega_{i-2}^i \sigma_i = \sigma_i \Omega_{i-1} \quad \text{and} \quad \Omega_{i-2}^i \tau_i = \tau_i \Omega_{i-1}.$$

This morphism induces a structural map

$$w_{i-2}^i : G_i \to G_i$$

of G for codimension 2 inverses. This map sends an i-arrow

$$x \underset{v}{\overset{u}{\Longrightarrow \alpha}} y$$

to an i-arrow

$$w_{i-2}^i(\alpha) : \quad y \underset{w_{i-2}^{i-1}(v)}{\overset{w_{i-2}^{i-1}(u)}{\Longrightarrow}} x \quad .$$

We will see that $w_{i-2}^i(\alpha)$ is an inverse of α for the composition $*_{i-2}^i$ up to an $(i+1)$-arrow.

• *Codimension* 1 *associativity constraints*

Let $i \geq 1$. The pair

$$\Big((\nabla_i \amalg_{D_{i-1}} 1_{D_i}) \nabla_i, (1_{D_i} \amalg_{D_{i-1}} \nabla_i) \nabla_i \Big) : D_i \to D_i \amalg_{D_{i-1}} D_i \amalg_{D_{i-1}} D_i$$

of morphism of C_1 is admissible. Indeed, if

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, : D_i \to D_i \amalg_{D_{i-1}} D_i \amalg_{D_{i-1}} D_i \quad \text{and} \quad \varepsilon_1', \varepsilon_2' : D_i \to D_i \amalg_{D_{i-1}} D_i$$

denote the canonical morphisms, then we have

$$\begin{aligned}
(\nabla_i \amalg_{D_{i-1}} D_i) \nabla_i \sigma_i &= (\nabla_i \amalg_{D_{i-1}} D_i) \varepsilon_2' \sigma_i = \varepsilon_3 \sigma_i \\
&= (\varepsilon_2, \varepsilon_3) \varepsilon_2' \sigma_i = (\varepsilon_2, \varepsilon_3) \nabla_i \sigma_i \\
&= (D_i \amalg_{D_i} \nabla_i) \varepsilon_2' \sigma_i \\
&= (D_i \amalg_{D_i} \nabla_i) \nabla_i \sigma_i.
\end{aligned}$$

In the same way, we get

$$(\nabla_i \amalg_{D_{i-1}} D_i) \nabla_i \tau_i = \varepsilon_1 \tau_i = (D_i \amalg_{D_i} \nabla_i) \nabla_i \tau_i.$$

Hence there exists a lifting

$$\alpha_i : D_{i+1} \to D_i \amalg_{D_{i-1}} D_i \amalg_{D_{i-1}} D_i$$

in C_2 such that

$$\alpha_i \sigma_{i+1} = (\nabla_i \amalg_{D_{i-1}} D_i) \nabla_i \quad \text{and} \quad \alpha_i \tau_{i+1} = (D_i \amalg_{D_{i-1}} \nabla_i) \nabla_i.$$

This morphism induces a structural map

$$a_i : G_i \times_{G_{i-1}} G_i \times_{G_{i-1}} G_i \to G_{i+1}$$

of G for associativity constraints for the composition $*_{i-1}^i$. This map sends i-arrows

$$\xrightarrow{\ u\ } \xrightarrow{\ v\ } \xrightarrow{\ w\ }$$

to an $(i+1)$-arrow

$$a_{w,v,u} : (w *_{i-1}^i v) *_{i-1}^i u \to w *_{i-1}^i (v *_{i-1}^i u).$$

This shows that the composition $*_{i-1}^i$ is associative up to $(i+1)$-arrows.

- *Codimension* 1 *unit constraints*

Let $i \geq 1$. Consider the pair

$$\left((1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \nabla_i, 1_{\mathrm{D}_i}\right) : \mathrm{D}_i \to \mathrm{D}_i$$

of morphisms of C_1. (Note that the fact that the morphism

$$(1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) : \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \to \mathrm{D}_i$$

is well-defined requires a calculation. We will skip these calculations in this section.)
We claim that this pair is admissible. Indeed, if $\varepsilon_1, \varepsilon_2 : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ denote the canonical morphisms, then we have

$$(1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \nabla_i \sigma_i = (1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \varepsilon_2 \sigma_i = \sigma_i \kappa_{i-1} \sigma_i = \sigma_i$$

and

$$(1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \nabla_i \tau_i = (1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \varepsilon_1 \tau_i = \tau_i.$$

Hence there exists a lifting

$$\rho_i : \mathrm{D}_{i+1} \to \mathrm{D}_i$$

in C_2 such that

$$\rho_i \sigma_{i+1} = (1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \nabla_i \quad \text{and} \quad \rho_i \tau_{i+1} = 1_{\mathrm{D}_i}.$$

This morphism induces a structural map

$$r_i : G_i \to G_{i+1}$$

of G for right unit constraints for k_{i-1}. This map sends an i-arrow $u : x \to y$ to an $(i+1)$-arrow

$$r_u : u *_{i-1}^i k_{i-1}(x) \to u.$$

This shows that $k_{i-1}(x)$ is a right unit for the composition $*_{i-1}^i$ up to an $(i+1)$-arrow.
We get in a similar way a morphism

$$\lambda_i : \mathrm{D}_{i+1} \to \mathrm{D}_i$$

of C_2 inducing a codimension 1 left unit constraint

$$l_i : G_i \to G_{i+1}.$$

We thus get an $(i+1)$-arrow

$$l_u : k_{i-1}(y) *_{i-1}^i u \to u.$$

showing that $k_{i-1}(y)$ is a left unit for the composition $*_{i-1}^i$ up to an $(i+1)$-arrow.

- *Codimension* 1 *inverse constraints*

Let $i \geq 1$. The pair

$$\left((1_{\mathrm{D}_i}, \Omega_i) \nabla_i, \tau_i \kappa_{i-1}\right) : \mathrm{D}_i \to \mathrm{D}_i$$

of morphism of C_1 is admissible. Indeed, if $\varepsilon_1, \varepsilon_2 : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ denote the canonical morphisms, then we have

$$(1_{\mathrm{D}_i}, \Omega_i)\nabla_i \sigma_i = (1_{\mathrm{D}_i}, \Omega_i)\varepsilon_2 \sigma_i = \Omega_i \sigma_i = \tau_i = \tau_i \kappa_{i-1} \sigma_i$$

and

$$(1_{\mathrm{D}_i}, \Omega_i)\nabla_i \tau_i = (1_{\mathrm{D}_i}, \Omega_i)\varepsilon_1 \tau_i = \tau_i = \tau_i \kappa_{i-1} \tau_i.$$

Hence there exists a lifting

$$\delta_i : \mathrm{D}_{i+1} \to \mathrm{D}_i$$

in C_2 such that

$$\delta_i \sigma_{i+1} = (1_{\mathrm{D}_i}, \Omega_i)\nabla_i \quad \text{and} \quad \delta_i \tau_{i+1} = \tau_i \kappa_{i-1}.$$

This morphism induces a structural map

$$d_i : G_i \to G_{i+1}$$

of G for right inverse constraints for w_{i-1}^i. This map sends an i-arrow $u : x \to y$ to an $(i+1)$-arrow

$$d_u : u *_{i-1}^i w_{i-1}^i(u) \to k_{i-1}(y).$$

This shows that $w_{i-1}^i(u)$ is a right inverse of u for the composition $*_{i-1}^i$ up to an $(i+1)$-arrow.

We get in a similar way a morphism

$$\gamma_i : \mathrm{D}_{i+1} \to \mathrm{D}_i$$

of C_2 inducing a left inverse constraint. We thus get an $(i+1)$-arrow

$$g_u : w_{i-1}^i(u) *_{i-1}^i u \to k_{i-1}(x)$$

showing that $w_{i-1}^i(u)$ is a left inverse of u for the composition $*_{i-1}^i$ up to an $(i+1)$-arrow.

3.8. Examples of morphisms appearing in C_3.

  • *Codimension 1 Mac Lane's pentagon constraints*
Let $i \geq 1$. Denote by

$$\varepsilon_1, \ldots, \varepsilon_4 : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$$

the canonical morphisms. Let $c_2 : \mathrm{D}_{i+1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ be the morphism

$$\Big((\mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \nabla_i)\alpha_i, (\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i)\alpha_i \Big)\nabla_{i+1}$$

and let $c_3 : \mathrm{D}_{i+1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ be the morphism

$$\Big((\varepsilon_1 \kappa_i, (\varepsilon_2, \varepsilon_3, \varepsilon_4)\alpha_i)\nabla_{i-1}^{i+1}, (\mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i)\alpha_i,$$
$$((\varepsilon_1, \varepsilon_2, \varepsilon_3)\alpha_i, \varepsilon_4 \kappa_i)\nabla_{i-1}^{i+1} \Big)\Big(\nabla_{i+1} \amalg_{\mathrm{D}_i} \mathrm{D}_{i+1} \Big)\nabla_{i+1}.$$

The pair

$$(c_3, c_2) : \mathrm{D}_{i+1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$$

of morphisms of C_2 is admissible. Indeed, if $\varepsilon_1', \varepsilon_2' : \mathrm{D}_{i+1} \to \mathrm{D}_{i+1} \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_{i+1}$ denote the canonical morphisms, then we have

$$
\begin{aligned}
c_3 \sigma_{i+1} &= \big((\varepsilon_1, \varepsilon_2, \varepsilon_3)\alpha_i, \varepsilon_4 \kappa_i\big) \nabla_{i-1}^{i+1} \sigma_{i+1} \\
&= \big((\varepsilon_1, \varepsilon_2, \varepsilon_3)\alpha_i, \varepsilon_4 \kappa_i\big)\big(\varepsilon_1' \sigma_{i+1}, \varepsilon_2' \sigma_{i+1}\big) \nabla_i \\
&= \big((\varepsilon_1, \varepsilon_2, \varepsilon_3)\alpha_i \sigma_{i+1}, \varepsilon_4 \kappa_i \sigma_{i+1}\big) \nabla_i \\
&= \big((\varepsilon_1, \varepsilon_2, \varepsilon_3)(\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i) \nabla_i, \varepsilon_4\big) \nabla_i \\
&= \big(\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i\big)\big(\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i\big) \nabla_i
\end{aligned}
$$

and

$$
\begin{aligned}
c_2 \sigma_{i+1} &= \big(\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i\big)\alpha_i \sigma_{i+1} \\
&= \big(\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i\big)\big(\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i\big) \nabla_i.
\end{aligned}
$$

A similar calculation shows that

$$
c_3 \tau_{i+1} = \big(\mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \nabla_i\big)\big(\mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \nabla_i\big) \nabla_i = c_2 \tau_{i+1}.
$$

Hence there exists a lifting

$$
\pi_i : \mathrm{D}_{i+2} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i
$$

in C_3 such that

$$
\pi_i \sigma_{i+2} = c_3 \quad \text{and} \quad \pi_i \tau_{i+2} = c_2.
$$

This morphism induces a structural map of G for Mac Lane's pentagon constraints for compositions $*_{i-1}^i$, $*_i^{i+1}$ and $*_{i-1}^{i+1}$. This map sends i-arrows

$$
\xrightarrow{\ u\ } \xrightarrow{\ v\ } \xrightarrow{\ w\ } \xrightarrow{\ x\ }
$$

to an $(i+2)$-arrow

- *Codimension* 1 *exchange constraints*

Let $i \geq 2$. Consider the pair

$$
\Big((\nabla_{i-2}^i \amalg_{\nabla_{i-1}} \nabla_{i-2}^i)\nabla_i, (\nabla_i \amalg_{\mathrm{D}_{i-2}} \nabla_i)\nabla_{i-2}^i\Big) : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-2}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i
$$

of morphisms of C_2. (Note that the left morphism is not a globular sum. Nevertheless, it is not hard to prove by hand that this sum exists. Its existence also follows from the fact that it is a generalized globular sum in the sense of Section 2.5 of [1]). We claim that this pair is admissible. Indeed, if $\varepsilon_1, \ldots, \varepsilon_4 : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-2}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ and $\varepsilon_1', \varepsilon_2' : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-2}} \mathrm{D}_i$ denote the canonical morphisms, then we have

$$
\begin{aligned}
(\nabla_{i-2}^i \amalg_{\nabla_{i-1}} \nabla_{i-2}^i) \nabla_i \sigma_i &= ((\varepsilon_1, \varepsilon_3) \nabla_{i-2}^i, (\varepsilon_2, \varepsilon_4) \nabla_{i-2}^i) \nabla_i \sigma_i \\
&= (\varepsilon_2, \varepsilon_4) \nabla_{i-2}^i \sigma_i \\
&= (\varepsilon_2, \varepsilon_4)(\varepsilon_1' \sigma_i, \varepsilon_2' \sigma_i) \nabla_{i-1} \\
&= (\varepsilon_2 \sigma_i, \varepsilon_4 \sigma_i) \nabla_{i-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_i \amalg_{\mathrm{D}_{i-2}} \nabla_i) \nabla_{i-2}^i \sigma_i &= (\nabla_i \amalg_{\mathrm{D}_{i-2}} \nabla_i)(\varepsilon_1' \sigma_i, \varepsilon_2' \sigma_i) \nabla_{i-1} \\
&= ((\varepsilon_1, \varepsilon_2) \nabla_i \sigma_i, (\varepsilon_3, \varepsilon_4) \nabla_i \sigma_i) \nabla_{i-1} \\
&= (\varepsilon_2 \sigma_i, \varepsilon_4 \sigma_i) \nabla_{i-1}.
\end{aligned}
$$

A similar calculation shows that

$$
\left(\nabla_{i-2}^i \amalg_{\nabla_{i-1}} \nabla_{i-2}^i\right) \nabla_i \tau_i = \left(\varepsilon_1 \tau_i, \varepsilon_3 \tau_i\right) \nabla_{i-1} = \left(\nabla_i \amalg_{\mathrm{D}_{i-2}} \nabla_i\right) \nabla_{i-2}^i \tau_i.
$$

Hence there exists a lifting

$$
\varepsilon_i : \mathrm{D}_{i+1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-2}} \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i
$$

in C_3 such that

$$
\varepsilon_i \sigma_{i+1} = \left(\nabla_{i-2}^i \amalg_{\nabla_{i-1}} \nabla_{i-2}^i\right) \nabla_i \quad \text{and} \quad \varepsilon_i \tau_{i+1} = \left(\nabla_i \amalg_{\mathrm{D}_{i-2}} \nabla_i\right) \nabla_{i-2}^i.
$$

This morphism induces a structural map of G for exchange constraints for compositions $*_{i-2}^{i-1}$, $*_{i-1}^i$ and $*_{i-2}^i$. This map sends i-arrows
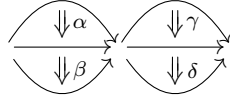
to an $(i+1)$-arrow

$$
e_{\gamma, \delta, \beta.\alpha} : \left(\delta *_{i-2}^i \beta\right) *_{i-1}^i \left(\gamma *_{i-2}^i \alpha\right) \to \left(\delta *_{i-1}^i \gamma\right) *_{i-2}^i \left(\beta *_{i-1}^i \alpha\right).
$$

This shows that the pair of compositions $(*_{i-2}^i, *_{i-1}^i)$ satisfies the exchange law up to $(i+1)$-arrows.

 • *Codimension 1 triangle constraints*

Let $i \geq 1$. Denote by $\varepsilon_1, \varepsilon_2 : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ the canonical morphisms. Let $d_2 : \mathrm{D}_{i+1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ be the morphism

$$
((\varepsilon_1 \kappa_i, \varepsilon_2 \lambda_i) \nabla_{i-1}^{i+1}, (\varepsilon_1, \varepsilon_1 \sigma_i \kappa_{i-1}, \varepsilon_2) \alpha_i) \nabla_{i+1}
$$

and let $d_1 : \mathrm{D}_{i+1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ be the morphism

$$(\varepsilon_1 \rho_i, \varepsilon_2 \kappa_i) \nabla_{i-1}^{i+1}.$$

The pair

$$(d_2, d_1) : \mathrm{D}_{i+1} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$$

of morphisms of C_2 is admissible. Indeed, we have

$$\begin{aligned}
d_2 \sigma_{i+1} &= (\varepsilon_1, \varepsilon_1 \sigma_i \kappa_{i-1}, \varepsilon_2) \alpha_i \sigma_{i+1} \\
&= (\varepsilon_1, \varepsilon_1 \sigma_i \kappa_{i-1}, \varepsilon_2)(\nabla_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i) \nabla_i \\
&= ((\varepsilon_1, \varepsilon_1 \sigma_i \kappa_{i-1}) \nabla_i, \varepsilon_2) \nabla_i \\
&= \big(\varepsilon_1 (1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \nabla_i), \varepsilon_2 \big) \nabla_i
\end{aligned}$$

and

$$\begin{aligned}
d_1 \sigma_{i+1} &= (\varepsilon_1 \rho_i \sigma_{i+1}, \varepsilon_2 \kappa_i \sigma_{i+1}) \nabla_i \\
&= \big(\varepsilon_1 (1_{\mathrm{D}_i}, \sigma_i \kappa_{i-1}) \nabla_i, \varepsilon_2 \big) \nabla_i.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
d_2 \tau_{i+1} &= (\varepsilon_1 \kappa_i, \varepsilon_2 \lambda_i) \nabla_{i-1}^{i+1} \tau_{i+1} \\
&= (\varepsilon_1 \kappa_i \tau_{i+1}, \varepsilon_2 \lambda_i \tau_{i+1}) \nabla_i \\
&= (\varepsilon_1, \varepsilon_2) \nabla_i \\
&= \nabla_i
\end{aligned}$$

and

$$\begin{aligned}
d_1 \tau_{i+1} &= (\varepsilon_1 \rho_i \tau_{i+1}, \varepsilon_2 \kappa_i \tau_{i+1}) \nabla_i \\
&= (\varepsilon_1, \varepsilon_2) \nabla_i \\
&= \nabla_i.
\end{aligned}$$

Hence there exists a lifting

$$\nu_i : \mathrm{D}_{i+2} \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$$

in C_3 such that

$$\nu_i \sigma_{i+2} = d_2 \quad \text{and} \quad \nu_i \tau_{i+2} = d_1.$$

This morphism induces a structural map of G for triangle constraints for compositions $*_{i-1}^i$, $*_i^{i+1}$ and $*_{i-1}^{i+1}$. This map sends i-arrows

$$x \xrightarrow{u} y \xrightarrow{v} z$$

to an $(i+2)$-arrow

One can show in a similar way that there exists in C_3 morphisms corresponding to the following structural maps:

- codimension 3 compositions;

- codimension 3 inverses;
- codimension 2 associativity constraints;
- codimension 2 unit constraints;
- codimension 2 inverse constraints.

Remark 3.9. The structural maps we have defined can be used to truncate G to a bicategory in which every arrow is weakly invertible (see Remark 4.7 for more details).

3.10. Examples of morphisms appearing in higher C_n's. One can show that there exists in C_4 morphisms corresponding to the following structural maps:

- codimension 4 compositions;
- codimension 4 inverses;
- codimension 3 associativity constraints;
- codimension 3 unit constraints;
- codimension 3 inverse constraints;
- codimension 2 Mac Lane's pentagon constraints;
- codimension 2 exchange constraints;
- codimension 2 triangle constraints;
- codimension 1 constraints on constraints appearing in C_3 (i.e., axioms for tricategories).

In general, in C_n we have

- codimension n compositions;
- codimension n inverses;
- for every k such that $1 \le k < n$, codimension $n - k$ constraints on constraints appearing in C_k.

3.11. Existence of pregroupoidal structures. Let C be a contractible globular extension. From our previous analysis, one easily obtains that C can be endowed (in a non-canonical way) with the structure of a *pregroupoidal globular extension* in the sense of [2], that is, with morphisms

$$\nabla^i_j : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_j} \mathrm{D}_i, \qquad i > j \ge 0,$$
$$\kappa_i : \mathrm{D}_{i+1} \to \mathrm{D}_i, \qquad i \ge 0,$$
$$\Omega^i_j : \mathrm{D}_i \to \mathrm{D}_i, \qquad i > j \ge 0,$$

such that

(1) for every i, j such that $i > j \ge 0$, we have

$$\nabla^i_j \sigma_i = \begin{cases} \varepsilon_2 \sigma_i, & j = i - 1, \\ (\sigma_i \amalg_{\mathrm{D}_j} \sigma_i) \nabla^{i-1}_j & j < i - 1, \end{cases}$$

and

$$\nabla^i_j \tau_i = \begin{cases} \varepsilon_1 \tau_i, & j = i - 1, \\ (\tau_i \amalg_{\mathrm{D}_j} \tau_i) \nabla^{i-1}_j & j < i - 1, \end{cases}$$

where $\varepsilon_1, \varepsilon_2 : \mathrm{D}_i \to \mathrm{D}_i \amalg_{\mathrm{D}_{i-1}} \mathrm{D}_i$ denote the canonical morphisms;

(2) for every $i \ge 0$, we have

$$\kappa_i \sigma_{i+1} = 1_{\mathrm{D}_i} \quad \text{and} \quad \kappa_i \tau_{i+1} = 1_{\mathrm{D}_i};$$

(3) for every i, j such that $i > j \geq 0$, we have

$$\Omega^i_j \sigma_i = \begin{cases} \tau_i & j = i - 1, \\ \sigma_i \Omega^{i-1}_j & j < i - 1, \end{cases}$$

and

$$\Omega^i_j \tau_i = \begin{cases} \sigma_i & j = i - 1, \\ \tau_i \Omega^{i-1}_j & j < i - 1. \end{cases}$$

Given a pregroupoidal globular extension structure on C, any ∞-groupoid G of type C is endowed with the structure of an ∞-*pregroupoids* in the sense of [2], that is, with maps

$$*^i_j : G_i \times_{G_j} G_i \to G_i, \quad i > j \geq 0,$$
$$k_i : G_i \to G_{i+1}, \quad i \geq 0,$$
$$w^i_j : G_i \to G_i, \quad i > j \geq 0,$$

such that

(1) for every (v, u) in $G_i \times_{G_j} G_i$ with $i > j \geq 0$, we have

$$s_i(v *^i_j u) = \begin{cases} s_i(u), & j = i - 1, \\ s_i(v) *^{i-1}_j s_i(u), & j < i - 1, \end{cases}$$

and

$$t_i(v *^i_j u) = \begin{cases} t_i(v), & j = i - 1, \\ t_i(v) *^{i-1}_j t_i(u), & j < i - 1; \end{cases}$$

(2) for every u in G_i with $i \geq 0$, we have

$$s_{i+1} k_i(u) = u = t_{i+1} k_i(u);$$

(3) for every u in G_i for $i \geq 1$ and j such that $i > j \geq 0$, we have

$$s_i(w^i_j(u)) = \begin{cases} t_i(u), & j = i - 1, \\ w^{i-1}_j(s_i(u)), & j < i - 1, \end{cases}$$

and

$$t_i(w^i_j(u)) = \begin{cases} s_i(u), & j = i - 1, \\ w^{i-1}_j(t_i(u)), & j < i - 1. \end{cases}$$

For $i \geq j \geq 0$, we will denote by k^j_i the map from $X_j \to X_i$ defined by

$$k^j_i = k_{i-1} \cdots k_{j+1} k_j.$$

4. Weak equivalences of ∞-groupoids

4.1. In this section, we fix a contractible globular extension C and an ∞-groupoid G of type C. Moreover, we *choose* once and for all a pregroupoidal globular extension structure on C. The ∞-groupoid G is thus endowed with the structure of an ∞-pregroupoid. We will use the same notation for the pregroupoidal structure on C and the structure of ∞-pregroupoid on G as in paragraph 3.11.

4.2. Homotopy relation between n-arrows. Let u and v be two n-arrows, $n \geq 0$, of G. A *homotopy* from u to v is an $(n+1)$-arrow from u to v. If such a homotopy exists, we will say that u is homotopic to v and we will write $u \sim_n v$. Note that if u is homotopic to v, then u and v are parallel.

Lemma 4.3. *For every $n \geq 0$, the relation $\sim_n$ is an equivalence relation. Moreover, if $n \geq 1$, this relation is compatible with the composition $*_{n-1}^n$.*

Proof. Let u be an n-arrow of G. The $(n+1)$-arrow $k_n(u)$ is a homotopy from u to u. The relation $\sim_n$ is hence reflexive.

Let now v be a second n-arrow of G and let $h : u \to v$ be a homotopy. The $(n+1)$-arrow $w_{n+1}(h)$ is a homotopy from v to u. The relation $\sim_n$ is hence symmetric.

Suppose now w is a third n-arrow of G and $k : v \to w$ is a second homotopy. Then the $(n+1)$-arrow $k *_n^{n+1} h$ is a homotopy from u to w. The relation $\sim_n$ is hence transitive.

Finally, suppose we have a diagram

in G, where single arrows are n-arrows with $n \geq 1$, and double arrows are $(n+1)$-arrows. The $(n+1)$-arrow $k *_{n-1}^{n+1} h$ is a homotopy from $v *_n^{n+1} u$ to $v' *_n^{n+1} u'$. The relation $\sim_n$ is hence compatible with the composition $*_{n-1}^n$. $\qquad\square$

4.4. The groupoid $\varpi_n(G)$. For $n \geq 0$, we will denote by $\overline{G_n}$ the quotient of G_n by the equivalence relation $\sim_n$.

Let us now fix $n \geq 1$. The maps
$$s_n, t_n : G_n \to G_{n-1}, \quad k_{n-1} : G_{n-1} \to G_n,$$
induce maps
$$s_n, t_n : \overline{G_n} \to G_{n-1}, \quad k_{n-1} : G_{n-1} \to \overline{G_n}.$$
Moreover, by the previous lemma, the map
$$*_{n-1}^n : G_n \times_{G_{n-1}} G_n \to G_n$$
induces a map
$$*_{n-1}^n : \overline{G_n} \times_{G_{n-1}} \overline{G_n} \to \overline{G_n}.$$
We will denote by $\varpi_n(G)$ the graph
$$\overline{G_n} \xrightarrow[t_n]{s_n} G_{n-1},$$
endowed with the maps
$$*_{n-1}^n : \overline{G_n} \times_{G_{n-1}} \overline{G_n} \to \overline{G_n} \quad \text{and} \quad k_{n-1} : G_{n-1} \to \overline{G_n}.$$

Proposition 4.5. *For every $n \geq 1$, $\varpi_n(G)$ is a groupoid.*

Proof. Let u, v and w be three $(n+1)$-arrows of G, composable in codimension 1. Choose, as in paragraph 3.7, a morphism
$$\alpha_n : D_{n+1} \to D_n \amalg_{D_{n-1}} D_n \amalg_{D_{n-1}} D_n$$

of C such that
$$\alpha_n \sigma_{n+1} = (\nabla_n \amalg_{D_{n-1}} D_n) \nabla_n \quad \text{and} \quad \alpha_n \tau_{n+1} = (D_n \amalg_{D_{n-1}} \nabla_n) \nabla_n.$$
This morphism induces an $(n+1)$-arrow
$$a_{w,v,u} : (w *_{n-1}^n v) *_{n-1}^n u \to w *_{n-1}^n (v *_{n-1}^n u),$$
thereby proving the associativity of the composition $*_{n-1}^n$ up to homotopy.

Let now $u : x \to y$ be an n-arrow. Choose, as in paragraph 3.7, morphisms
$$\lambda_n : D_{n+1} \to D_n \quad \text{and} \quad \rho_n : D_{n+1} \to D_n,$$
of C such that
$$\lambda_n \sigma_{n+1} = (\tau_n \kappa_{n-1}, 1_{D_n}) \nabla_n \quad \text{and} \quad \lambda_n \tau_{n+1} = 1_{D_n},$$
$$\rho_n \sigma_{n+1} = (1_{D_n}, \sigma_n \kappa_{n-1}) \nabla_n \quad \text{and} \quad \rho_n \tau_{n+1} = 1_{D_n}.$$
These morphisms induce $(n+1)$-arrows
$$l_u : k_{n-1}(y) *_{n-1}^n u \to u \quad \text{and} \quad r_u : u *_{n-1}^n k_{n-1}(x) \to u,$$
thereby proving that $k_{n-1}(x)$ is a unit up to homotopy.

Let us now prove that $w_n(u) : y \to x$ is an inverse of u up to homotopy. Choose, as in paragraph 3.7, morphisms
$$\gamma_n : D_{n+1} \to D_n \quad \text{and} \quad \delta_n : D_{n+1} \to D_n$$
of C such that
$$\gamma_n \sigma_{n+1} = (\Omega_n, 1_{D_n}) \nabla_n \quad \text{and} \quad \gamma_n \tau_{n+1} = \sigma_n \kappa_{n-1},$$
$$\delta_n \sigma_{n+1} = (1_{D_n}, \Omega_n) \nabla_n \quad \text{and} \quad \delta_n \tau_{n+1} = \tau_n \kappa_{n-1}.$$
These morphisms induce $(n+1)$-arrows
$$g_u : w_n(u) *_{n-1}^n u \to x \quad \text{and} \quad d_u : f *_{n-1}^n w_n(u) \to y,$$
thus ending the proof. $\qquad\square$

Proposition 4.6. *The groupoid $\varpi_n(G)$ does not depend on the choice of a pregroupoidal globular structure on C.*

Proof. The groupoid $\varpi_n(G)$ depends a priori of the choice of
$$\nabla_n : D_n \to D_n \amalg_{D_{n-1}} D_n \quad \text{and} \quad \kappa_{n-1} : D_n \to D_{n-1}.$$
Let us show it does not.

Let
$$\nabla'_n : D_n \to D_n \amalg_{D_{n-1}} D_n$$
be a morphism of C with same globular source and target as ∇_n. Denote by
$$*_{n-1}^{'n} : G_n \times_{G_{n-1}} G_n \to G_n$$
the induced composition. Since the pair $(\nabla_n, \nabla'_n) : D_n \to D_n \amalg_{D_{n-1}} D_n$ is admissible, there exists a lifting
$$\mu : D_{n+1} \to D_n \amalg_{D_{n-1}} D_n$$
in C such that
$$\mu \sigma_{n+1} = \nabla_n \quad \text{and} \quad \mu \tau_{n+1} = \nabla'_n.$$

If u and v are two n-arrows of G, composable in codimension 1, then μ induces an $(n+1)$-arrow

$$m_{v,u} : v *_{n-1}^n u \to v *_{n-1}'^n u.$$

Hence the independence from ∇_n.

In the same way, if $\kappa'_{n-1} : D_n \to D_{n-1}$ is a morphism of C with same globular source and target as κ_{n-1}, then the admissible pair $(\kappa_{n-1}, \kappa'_{n-1}) : D_n \to D_{n-1}$ admits a lifting from which we immediately get the independence from κ_{n-1}. $\qquad\square$

Remark 4.7. For any $n \geq 1$, $\varpi_n(G)$ is actually the truncation in dimension 1 of a bigroupoid $\varpi_n^2(G)$, i.e., of a bicategory in which every 1-arrow is invertible up to a 2-arrow and every 2-arrow is invertible. Let us briefly explain how to define this bigroupoid. The underlying 2-graph of $\varpi_n^2(G)$ is

$$\overline{G_{n+1}} \underset{t_{n+1}}{\overset{s_{n+1}}{\rightrightarrows}} G_n \underset{t_n}{\overset{s_n}{\rightrightarrows}} G_{n-1},$$

and its structure maps are induced by a choice of maps k_{n-1}, k_n, $*_{n-1}^n$, $*_{n-1}^{n+1}$, $*_n^{n+1}$, a_n, l_n and r_n as in the previous section. The axioms of bigroupoids follow from the existence in C of

- codimension 1 and 2 associativity constraints;
- codimension 1 and 2 unit constraints;
- codimension 1 pentagon constraints;
- codimension 1 triangle constraints;
- codimension 1 and 2 inverse constraints.

One can easily show that $\varpi_n^2(G)$ does not depend on the choice of the above maps (up to biequivalence).

The bigroupoid $\varpi_1^2(G)$ (resp. the groupoid $\varpi_1(G)$) can be thought of as a truncation of G in dimension 2 (resp. in dimension 1).

The author has no doubt that a reader more patient than him could check that a similar construction gives rise to a tricategory in the sense of [10] in which every arrow is weakly invertible.

4.8. The functor ϖ_n. Let $f : G \to H$ be a morphism of ∞-groupoids of type C. Such a morphism induces a morphism of globular sets between the underlying globular sets and in particular respects the notion of homotopy between n-arrows. It follows that for any $n \geq 1$, f induces a morphism of graphs

$$\varpi_n(f) : \varpi_n(G) \to \varpi_n(H).$$

The naturality of f implies that $\varpi_n(f)$ is actually a functor. We thus get a functor

$$\varpi_n : \infty\text{-}\mathcal{G}\mathrm{pd}_C \to \mathcal{G}\mathrm{pd},$$

where $\mathcal{G}\mathrm{pd}$ denotes the category of groupoids.

4.9. Inverse image of globular functors. If $F : D \to D'$ is a morphism of globular extensions, then the precomposition by F defines a functor $\mathrm{Mod}(D') \to \mathrm{Mod}(D)$ that we will denote by F^*. In particular, if D and D' are contractible, we get a functor $F^* : \infty\text{-}\mathcal{G}\mathrm{pd}_{D'} \to \infty\text{-}\mathcal{G}\mathrm{pd}_D$. Since F is a functor under $\mathbb{G}$, the underlying globular sets of an ∞-groupoid G of type D' and of $F^*(G)$ coincide.

Proposition 4.10. *Let $F : D \to D'$ be a morphism of contractible globular extensions. Then for any $n \geq 1$, the triangle*

$$\infty\text{-}\mathcal{G}\text{pd}_{D'} \xrightarrow{\ F^* \ } \infty\text{-}\mathcal{G}\text{pd}_D$$
$$\varpi_n \searrow \qquad \swarrow \varpi_n$$
$$\mathcal{G}\text{pd}$$

commutes.

Proof. Let G be an ∞-groupoid of type D'. Since G and $F^*(G)$ have the same underlying globular set, the underlying graphs of $\varpi_n(G)$ and $\varpi_n(F^*(G))$ coincide. The groupoid structure of $\varpi_n(F^*(G))$ is induced by a choice of morphisms

$$\nabla_n : D_n \to D_n \amalg_{D_{n-1}} D_n,$$
$$\kappa_{n-1} : D_n \to D_{n-1}$$

of D such that

$$\nabla_n \sigma_n = \varepsilon_2 \sigma_n \quad \text{and} \quad \nabla_n \tau_n = \varepsilon_1 \tau_n,$$
$$\kappa_{n-1} \sigma_n = 1_{D_{n-1}} \quad \text{and} \quad \kappa_{n-1} \tau_n = 1_{D_{n-1}},$$

where $\varepsilon_1, \varepsilon_2 : D_n \to D_n \amalg_{D_{n-1}} D_n$ denote the canonical morphisms. By applying F to these morphisms, we get morphisms

$$F(\nabla_n) : D_n \to D_n \amalg_{D_{n-1}} D_n,$$
$$F(\kappa_{n-1}) : D_n \to D_{n-1}$$

of D' such that

$$F(\nabla_n)\sigma_n = \varepsilon_2 \sigma_n \quad \text{and} \quad F(\nabla_n)\tau_n = \varepsilon_1 \tau_n,$$
$$F(\kappa_{n-1})\sigma_n = 1_{D_{n-1}} \quad \text{and} \quad F(\kappa_{n-1})\tau_n = 1_{D_{n-1}}.$$

These morphisms can be used to define the groupoid structure of $\varpi_n(G)$. (Note that we have neglected some inoffensive canonical isomorphisms when describing the target of $F(\nabla_n)$ and $F(\kappa_{n-1})$.) But it is clear that these two choices lead to the same groupoid structure. □

4.11. Homotopy groups of ∞-groupoids. We define the set $\pi_0(G)$ of *connected components* of G by

$$\pi_0(G) = \pi_0(\varpi_1(G)) = \overline{G_0}.$$

Let $n \geq 1$ and let u, v be two parallel $(n-1)$-arrows of G. We will denote by $\mathsf{Hom}_G(u, v)$ the set of n-arrows of G from u to v. We set

$$\pi_n(G, u, v) = \mathsf{Hom}_{\varpi_n(G)}(u, v) \quad \text{and} \quad \pi_n(G, u) = \pi_n(G, u, u).$$

The set $\pi_n(G, u, v)$ is nothing but the quotient of $\mathsf{Hom}_G(u, v)$ by the equivalence relation $\sim_n$. Note that $\pi_n(G, u)$ is canonically endowed with a group structure.

If $n \geq 1$ and x is an object of G, we define the *n-th homotopy group* of (G, x) as

$$\pi_n(G, x) = \pi_n(k^0_{n-1}(x)).$$

The Eckmann-Hilton argument shows that for $n \geq 2$, the group $\pi_n(G, x)$ is abelian.

From the fact that for every $n \geq 1$, ϖ_n is a functor from the category of ∞-groupoids of type C to groupoids, we get that

- π_0 is a functor from the category of ∞-groupoids of type C to the category of sets;
- for all $n \geq 1$, π_n is a functor from the category of ∞-groupoids of type C endowed with an $(n-1)$-arrow (or with an object) to the category of groups.

Lemma 4.12 (Division lemma). *Let $n \geq 2$ and let i be an integer such that $0 \leq i < n-1$. Let u, v be a pair of parallel $(n-1)$-arrows of G and let $\gamma : u' \to v'$ be an n-arrow such that*

$$s_i^n(\gamma) = t_i^{n-1}(u) = t_i^{n-1}(v).$$

Then the map

$$\mathsf{Hom}_G(u, v) \to \mathsf{Hom}_G(u' *_i^{n-1} u, v' *_i^{n-1} v)$$

$$\alpha \mapsto \gamma *_i^n \alpha$$

induces a natural bijection

$$\pi_n(G, u, v) \to \pi_n(G, u' *_i^{n-1} u, v' *_i^{n-1} v).$$

Proof. We will denote by K the map

$$\mathsf{Hom}_G(u, v) \to \mathsf{Hom}_G(u' *_i^{n-1} u, v' *_i^{n-1} v)$$

$$\alpha \mapsto \gamma *_i^n \alpha \quad .$$

If $\Lambda : \alpha \to \alpha'$ is a homotopy between two n-arrows $\alpha, \alpha' : u \to v$, then the $(n+1)$-arrow

$$k_n(\gamma) *_i^{n+1} \Lambda : \gamma *_i^n \alpha \to \gamma *_i^n \alpha'$$

is a homotopy from $K(\alpha)$ to $K(\alpha')$. Hence the map K induces a map

$$\overline{K} : \pi_n(G, u, v) \to \pi_n(G, u' *_i^{n-1} u, v' *_i^{n-1} v).$$

We will construct a map

$$L : \mathsf{Hom}_G(G, u' *_i^{n-1} u, v' *_i^{n-1} v) \to \mathsf{Hom}_G(u, v)$$

inducing an inverse

$$\overline{L} : \pi_n(G, u' *_i^{n-1} u, v' *_i^{n-1} v) \to \pi_n(G, u, v)$$

of $\overline{K}$.

Our proof will be quite technical. For this reason, we start by giving an idea of it. Let $\beta : u' *_i^{n-1} u \to v' *_i^{n-1} v$. One could naively think that $L'(\beta) = w_i^n(\gamma) *_i^n \beta$ would induce an inverse (that would be true if G were a strict ∞-groupoid). But the source (resp. the target) of $L'(\beta)$ is

$$w_i^{n-1}(u') *_i^{n-1} (u' *_i^{n-1} u) \qquad (\text{resp. } w_i^{n-1}(v') *_i^{n-1} (v' *_i^{n-1} v)).$$

In particular, $L'(\beta)$ does not belong to $\mathsf{Hom}_G(u, v)$. Nevertheless, the source (resp. the target) of $L'(\beta)$ and u (resp. of $L'(\beta)$ and v) coincide in dimension i, that is, we have

$$s_i^n(L'(\beta)) = s_i^{n-1}(u) \quad \text{and} \quad t_i^n(L'(\beta)) = t_i^{n-1}(v).$$

We will "correct" the source and target of $L'(\beta)$ dimension by dimension. We will first construct $(i+2)$-arrows c_{i+2} and d_{i+2} such that the n-arrow

$$\alpha_{i+2} = k_n^{i+2}(d_{i+2}) *_{i+1}^n (L'(\beta) *_{i+1}^n k_n^{i+2}(c_{i+2}))$$

has the same source (resp. the same target) as u (resp. as v) in dimension $i + 1$. By induction, we will define α_j for $i < j \leq n$ such that the source (resp. the target) of α_j and u (resp. of α_j and v) coincide in dimension $j - 1$ (α_{i+1} being $L'(\beta)$). We will end up by defining $L(\beta)$ as the n-arrow

$$\alpha_n = d_n *_{n-1}^n \left[\left[\cdots \left(k_n^{i+2}(d_{i+2}) *_{i+1}^n \left(L'(\beta) *_{i+1}^n k_n^{i+2}(c_{i+2}) \right) \right) \cdots \right] *_{n-1}^n c_n \right],$$

where the c_j's (resp. the d_j's) are the j-arrows "correcting the source (resp. the target) of $L'(\beta)$ in dimension $j - 1$."

Here is how our proof is organized. First, we define the c_j's (resp. the d_j's) as functions C_j (resp. D_j) of u and u' (resp. of v and v'). To define these functions, we will define by mutual induction (on j) the C_j's (resp. the D_j's), their sources C_j^- (resp. D_j^-), their targets C_j^+ (resp. D_j^+) and functions S_j (resp. T_j) which (as will be proved in the fourth step) are the sources (resp. the targets) of the α_j's. An important point is that all these functions come from morphisms of the contractible globular extension C. This will allow us to get liftings from C. Second, we show that the pairs of morphisms of C inducing the pairs (C_j^-, C_j^+) and (D_j^-, D_j^+) are admissible. This is actually needed by the induction step of the first point. Third, we define the α_j's and we prove that their sources (resp. their targets) are given by the S_j's (resp. the T_j's). Fourth, we define L and $\overline{L}$. Fifth, we show that $\overline{L}$ is an inverse of $\overline{K}$.

1. *Definition of the C_j's and the D_j's*

We define by induction on j such that $i + 2 \leq j \leq n$ maps

$$S_j, T_j : G_{n-1} \times_{G_i} G_{n-1} \to G_{n-1},$$
$$C_j^-, C_j^+, D_j^-, D_j^+ : G_{n-1} \times_{G_i} G_{n-1} \to G_{j-1},$$
$$C_j, D_j : G_{n-1} \times_{G_i} G_{n-1} \to G_j,$$

induced by morphisms of C.

We set

$$S_{i+2}(u', u) = w_i^{n-1}(u') *_i^{n-1} (u' *_i^{n-1} u),$$
$$T_{i+2}(v', v) = w_i^{n-1}(v') *_i^{n-1} (v' *_i^{n-1} v);$$

for j such that $i + 2 < j \leq n$, we set

$$S_j(u', u) = k_{n-1}^{j-1} D_{j-1}(u', u) *_{j-2}^{n-1} (S_{j-1}(u', u) *_{j-2}^{n-1} k_{n-1}^{j-1} C_{j-1}(u', u)),$$
$$T_j(v', v) = k_{n-1}^{j-1} D_{j-1}(v', v) *_{j-2}^{n-1} (T_{j-1}(v', v) *_{j-2}^{n-1} k_{n-1}^{j-1} C_{j-1}(v', v));$$

for j such that $i + 2 \leq j \leq n$, we set

$$C_j^-(u', u) = s_{j-1}^{n-1}(u),$$
$$C_j^+(u', u) = s_{j-1}^{n-1} S_j(u', u),$$
$$D_j^-(v', v) = s_{j-1}^{n-1} T_j(v', v),$$
$$D_j^+(v', v) = t_{j-1}^{n-1}(v).$$

For our definition to be complete, we need to define the C_j's and the D_j's. Let j be such that $i + 2 \leq j \leq n$. By induction, the maps

$$C_j^-, C_j^+ : G_{n-1} \times_{G_i} G_{n-1} \to G_{j-1}$$

are induced by a pair of morphisms of C. We will prove in the second step of the proof that this pair is admissible. Admitting this fact, we get a lifting of the pair and so a map

$$C_j : G_{n-1} \times_{G_i} G_{n-1} \to G_j$$

such that

$$s_j C_j = C_j^- \quad \text{and} \quad t_j C_j = C_j^+.$$

In the same way, we get a map

$$D_j : G_{n-1} \times_{G_i} G_{n-1} \to G_j$$

induced by C, such that

$$s_j D_j = D_j^- \quad \text{and} \quad t_j D_j = D_j^+.$$

2. *The pairs (C_j^-, C_j^+) and (D_j^-, D_j^+) are admissible*

Let us check that the pair of morphisms of C inducing the pair (C_j^-, C_j^+) is admissible. It suffices to check that the $(j-1)$-arrows $C_j^-(u', u)$ and $C_j^+(u', u)$ are parallel for every (u', u) in $G_{n-1} \times_{G_i} G_{n-1}$. (The reader not convinced by this assertion can either extract a direct proof by dualizing our calculations, or read paragraph 5.3 of [2].) Let (u', u) be in $G_{n-1} \times_{G_i} G_{n-1}$. For $j = i + 2$, we have

$$
\begin{aligned}
s_{i+1} C_{i+2}^+(u', u) &= s_{i+1} s_{i+1}^{n-1} S_{i+2}(u', u) \\
&= s_i^{n-1}(w_i^{n-1}(u') *_i^{n-1} (u' *_i^{n-1} u)) \\
&= s_i^{n-1}(u) = s_{i+1} s_{i+1}^{n-1}(u) \\
&= s_{i+1} C_{i+2}^-(u', u)
\end{aligned}
$$

and

$$
\begin{aligned}
t_{i+1} C_{i+2}^+(u', u) &= t_{i+1} s_{i+1}^{n-1} S_j(u', u) \\
&= t_i^{n-1}(w_i^{n-1}(u') *_i^{n-1} (u' *_i^{n-1} u)) \\
&= t_i^{n-1} w_i^{n-1}(u') \\
&= s_i^{n-1}(u') = t_i^{n-1}(u) = t_{i+1} s_{i+1}^{n-1}(u) \\
&= t_{i+1} C_{i+2}^-(u', u);
\end{aligned}
$$

and for $i + 2 < j \leq n$, we have

$$
\begin{aligned}
s_{j-1} C_j^+(u', u) &= s_{j-1} s_{j-1}^{n-1} S_j(u', u) \\
&= s_{j-2}^{n-1}(k_{n-1}^{j-1} D_{j-1}(u', u) *_{j-2}^{n-1} (S_{j-1}(u', u) *_{j-2}^{n-1} k_{n-1}^{j-1} C_{j-1}(u', u))) \\
&= s_{j-2}^{n-1} k_{n-1}^{j-1} C_{j-1}(u', u) = s_{j-1} C_{j-1}(u', u) \\
&= C_{j-1}^-(u', u) = s_{j-2}^{n-1}(u) = s_{j-1} s_{j-1}^{n-1}(u) \\
&= s_{j-1} C_j^-(u', u)
\end{aligned}
$$

and

$$
\begin{aligned}
t_{j-1}C_j^+(u',u) &= t_{j-1}s_{j-1}^{n-1}S_j(u',u) \\
&= t_{j-2}^{n-1}(k_{n-1}^{j-1}D_{j-1}(u',u) *_{j-2}^{n-1}(S_{j-1}(u',u) *_{j-2}^{n-1}k_{n-1}^{j-1}C_{j-1}(u',u))) \\
&= t_{j-2}^{n-1}k_{n-1}^{j-1}D_{j-1}(u',u) = t_{j-1}D_{j-1}(u',u) \\
&= D_{j-1}^+(u',u) = t_{j-2}^{n-1}(u) = t_{j-1}s_{j-1}^{n-1}(u) \\
&= t_{j-1}C_j^-(u',u).
\end{aligned}
$$

Very similar calculations show that the $(j-1)$-arrows $D_j^-(v',v)$ and $D_j^+(v',v)$ are parallel for every (v',v) in $G_{n-1}\times_{G_i}G_{n-1}$.

3. *Definition of the α_j's and calculation of their sources and targets*

Let $\beta: u' *_i^{n-1} u \to v' *_i^{n-1} v$. We define by induction on j such that $i < j \le n$, an n-arrow α_j. For $j = i+1$, we set

$$
\alpha_{i+1} = w_i^n(\gamma) *_i^n \beta.
$$

For $j > i+1$, we set

$$
\alpha_j = k_n^j(d_j) *_{j-1}^n (\alpha_{j-1} *_{j-1}^n k_n^j(c_j)),
$$

where

$$
c_j = C_j(u',u) \quad \text{and} \quad d_j = D_j(v',v).
$$

To show that our α_j's are well-defined, we have to prove that

$$
c_j : s_{j-1}^{n-1}(u) \to s_{j-1}^n(\alpha_{j-1}) \quad \text{and} \quad d_j : t_{j-1}^n(\alpha_{j-1}) \to t_{j-1}^{n-1}(v).
$$

We first show by induction on j such that $i+1 < j \le n$ that we have

$$
s_n(\alpha_{j-1}) = S_j(u,u').
$$

For $j = i+2$, we have

$$
\begin{aligned}
s_n(\alpha_{i+1}) &= s_n(w_i^n(\gamma) *_i^n \beta) \\
&= s_n w_i^n(\gamma) *_i^{n-1} s_n(\beta) \\
&= w_i^{n-1}s_n(\gamma) *_i^{n-1} s_n(\beta) \\
&= w_i^{n-1}(u') *_i^{n-1} (u' *_i^{n-1} u) \\
&= S_{i+2}(u',u);
\end{aligned}
$$

and for $j > i+1$, we have

$$
\begin{aligned}
s_n(\alpha_{j-1}) &= s_n(k_n^{j-1}(d_{j-1}) *_{j-2}^n (\alpha_{j-2} *_{j-2}^n k_n^{j-1}(c_{j-1}))) \\
&= (k_{n-1}^{j-1}(d_{j-1}) *_{j-2}^{n-1}(s_n\alpha_{j-2} *_{j-2}^n k_{n-1}^{j-1}(c_{j-1}))) \\
&= k_{n-1}^{j-1}D_{j-1}(u',u) *_{j-2}^{n-1}(S_{j-1}(u',u) *_{j-2}^{n-1}k_{n-1}^{j-1}C_{j-1}(u',u)) \\
&= S_j(u',u).
\end{aligned}
$$

We hence have

$$
\begin{aligned}
s_j(c_j) &= s_j C_j(u',u) = C_j^-(u',u) \\
&= s_{j-1}^{n-1}(u)
\end{aligned}
$$

and

$$t_j(c_j) = t_j C_j(u', u) = C_j^+(u', u)$$
$$= s_{j-1}^{n-1} S_j(u', u) = s_{j-1}^{n-1} s_n(\alpha_{j-1})$$
$$= s_{j-1}^n(\alpha_{j-1}),$$

i.e.,

$$c_j : s_{j-1}^{n-1}(u) \to s_{j-1}^n(\alpha_{j-1}).$$

Very similar calculations give

$$d_j : t_{j-1}^n(\alpha_{j-1}) \to t_{j-1}^{n-1}(v).$$

4. *Definition of L and $\overline{L}$*

Let $\beta : u' *_i^{n-1} u \to v' *_i^{n-1} v$ and let $\alpha = \alpha_n$ be the n-arrow defined in the previous step. Explicitly, we have

$$\alpha = d_n *_{n-1}^n \left[\left[\cdots \left(k_n^{i+2}(d_{i+2}) *_{i+1}^n \left(\left(w_i^n(\gamma) *_i^n \beta \right) *_{i+1}^n k_n^{i+2}(c_{i+2}) \right) \right) \cdots \right] *_{n-1}^n c_n \right].$$

Note that

$$s_n(\alpha) = s_n(c_n) = u,$$
$$t_n(\alpha) = t_n(d_n) = v.$$

We can thus define the map

$$L : \mathsf{Hom}_G(G, u' *_i^{n-1} u, v' *_i^{n-1} v) \to \mathsf{Hom}_G(u, v)$$

by sending β to α. The formula defining α is clearly functorial in β and the map L thus induces a map

$$\overline{L} : \pi_n(G, u' *_i^{n-1} u, v' *_i^{n-1} v) \to \pi_n(G, u, v).$$

5. *The map $\overline{L}$ is an inverse of $\overline{K}$*

We first prove that $\overline{L}$ is a left inverse of $\overline{K}$. Consider the maps

$$G_n \times_{G_i} G_n \to G_n$$
$$(\alpha, \gamma) \mapsto \alpha,$$
$$(\alpha, \gamma) \mapsto LK(\alpha).$$

They are both induced by a morphism of C. Moreover, we already know that α and $LK(\alpha)$ are parallel. We thus get from C an $(n+1)$-arrow from α to $LK(\alpha)$, thereby proving that we have

$$\overline{L}\,\overline{K} = 1_{\pi_n(G, u, v)}.$$

Let us now show that $\overline{L}$ is injective. We have just shown that $\overline{K}$ is injective. This means that for every n-arrow δ and all j such that $0 \leq j < n-1$, the right composition by δ in codimension $n-j$ (i.e., the operation $\alpha \mapsto \alpha *_j^n \delta$) reflects the property of being homotopic. Dually, the left composition by δ in codimension $n-j$ (i.e., the operation $\alpha \mapsto \delta *_j^n \alpha$) reflects the property of being homotopic. But the map L is obtained as a composition of such operations. Hence the map L also reflects the property of being homotopic. This exactly means that $\overline{L}$ is injective.

It follows that $\overline{L}$ is bijective. Its left inverse $\overline{K}$ is hence an inverse. $\square$

Theorem 4.13. *Let $n \geq 1$ and let u be an $(n-1)$-arrow of G. Set $x = s_0^{n-1}(u)$. There exists an isomorphism*

$$\pi_n(G, u) \to \pi_n(G, x),$$

natural in G.

Proof. If $n = 1$, the result is tautological. For $n \geq 2$, by the previous lemma and its dual, we have the following zig-zag of natural isomorphisms:

$$\pi_n(G, u) \overset{\sim}{\to} \pi_n(G, u *_0^{n-1} k_{n-1}^0(x)) \overset{\sim}{\leftarrow} \pi_n(G, k_{n-1}^0(x)) = \pi_n(G, x).$$

$\square$

Corollary 4.14. *Let $n \geq 1$ and let $u : x \to y$ be a 1-arrow. Then u induces an isomorphism*

$$\pi_n(G, x) \to \pi_n(G, y).$$

Proof. This follows from the previous theorem and its dual applied to $k_{n-1}^1(u)$. $\square$

Corollary 4.15. *Let $n \geq 1$ and let x be an object of G. The group $\pi_n(G, x)$ does not depend on the choice of the pregroupoidal globular structure on C.*

Proof. This follows from the previous theorem and Proposition 4.6 (i.e., the analogous result for $\pi_n(G, u)$). $\square$

Remark 4.16. We have only proved that $\pi_n(G, x)$ is unique up to a non-canonical isomorphism. Indeed, the zig-zag appearing in the proof of Theorem 4.13 depends on $*_0^{n-1}$, and hence on the choice of ∇_0^{n-1}. Nevertheless, one can show that the isomorphism of the theorem does not depend on this choice (the proof of this fact is very similar to the one of Lemma 4.12). We hence obtain that $\pi_n(G, x)$ is unique up to a canonical isomorphism.

4.17. Weak equivalences of ∞-groupoids. We will say that a morphism $f : G \to H$ of ∞-groupoids of type C is a *weak equivalence* if the following conditions are satisfied:

- the map $\pi_0(f) : \pi_0(G) \to \pi_0(H)$ is a bijection;
- for all $n \geq 1$ and every object x of G, the morphism $\pi_n(G, x) \to \pi_n(H, f(x))$, induced by f, is an isomorphism.

By the previous corollary, this definition does not depend on the choice of a pregroupoidal globular structure on C.

Theorem 4.18. *Let $f : G \to H$ be a morphism of ∞-groupoids of type C. The following conditions are equivalent:*

(1) *f is a weak equivalence;*
(2) *the map*

$$\pi_0(f) : \pi_0(G) \to \pi_0(H)$$

is a bijection, and for all $n \geq 1$ and every $(n-1)$-arrow u of G, the morphism f induces an isomorphism of groups

$$\pi_n(G, u) \to \pi_n(G, f(u));$$

(3) *the functor*

$$\varpi_1(f) : \varpi_1(G) \to \varpi_1(H)$$

is an equivalence of categories, and for every $n \geq 2$ and every pair u, v of parallel $(n-1)$-arrows of G, the morphism f induces a bijection

$$\pi_n(G, u, v) \to \pi_n(G, f(u), f(v));$$

(4) *the functor*

$$\varpi_1(f) : \varpi_1(G) \to \varpi_1(H)$$

is full and essentially surjective, and for every $n \geq 2$ and every pair u, v of parallel $(n-1)$-arrows of G, the morphism f induces a surjection

$$\pi_n(G, u, v) \to \pi_n(G, f(u), f(v)).$$

Proof. The equivalence of (1) and (2) is an immediate consequence of the previous theorem.

The implication $(3) \Rightarrow (2)$ is obvious. Let us show the reciprocal. Let $n \geq 1$ and let u, v be two parallel $(n-1)$-arrows of G. Suppose there exists an n-arrow $\alpha : u \to v$ in G and consider the map

$$\pi_n(G, u) \to \pi_n(G, u, v),$$

which sends the n-arrow β to the n-arrow $\alpha *_{n-1}^n \beta : u \to v$. Since $\varpi_n(G)$ is a groupoid, this map is a bijection. The morphism f obviously commutes with this isomorphism, that is, the square

$$
\begin{array}{ccc}
\pi_n(G, u) & \longrightarrow & \pi_n(G, u, v) \\
\downarrow & & \downarrow \\
\pi_n(H, f(u))) & \longrightarrow & \pi_n(H, f(u), f(v))
\end{array}
$$

is commutative. By hypothesis, the bottom horizontal arrow is a bijection and it follows that the top horizontal arrow is also a bijection. Thus, it suffices to show that if there exists an n-arrow $\beta : f(u) \to f(v)$ in H, then there exists an n-arrow $\alpha : u \to v$ in G. This is obvious when $n = 1$ by injectivity of $\pi_0(f)$. So let $n \geq 2$ and let $\beta : f(u) \to f(v)$ be an n-arrow of H. Set $x = s_{n-1}(u)$ and $y = t_{n-1}(v)$. The arrow $k_{n-1}(w_{n-1}(f(u))) *_{n-2}^n \beta$ is an n-arrow of H from $w_{n-1}(f(u)) *_{n-2}^{n-1} f(u) : f(x) \to f(x)$ to $w_{n-1}(f(u)) *_{n-2}^{n-1} f(v) : f(x) \to f(x)$. By injectivity of the map

$$\pi_{n-1}(G, x) \to \pi_{n-1}(H, f(x)),$$

the $(n-1)$-arrows $w_{n-1}(u) *_{n-2}^{n-1} u$ and $w_{n-1}(u) *_{n-2}^{n-1} v$ are equal in $\pi_{n-1}(G, x)$. Since $\varpi_{n-1}(G)$ is a groupoid, this implies that $u = v$ in $\pi_{n-1}(G, x, y)$ and so that there exists an n-arrow $\alpha : u \to v$.

The implication $(3) \Rightarrow (4)$ is obvious. Let us show the reciprocal. Let $n \geq 1$. Let u, v be two parallel $(n-1)$-arrows of G and let α, β be two n-arrows from u to v. Suppose we have $f(\alpha) = f(\beta)$ in $\pi_n(H, f(u), f(v))$. By definition, there exists an $(n+1)$-arrow of H from $f(\alpha)$ to $f(\beta)$. By surjectivity of the map

$$\pi_{n+1}(G, \alpha, \beta) \to \pi_{n+1}(H, f(\alpha), f(\beta)),$$

there exists an $(n+1)$-arrow of G from α to β, and we thus have $\alpha = \beta$ in $\pi_n(G, u, v)$. $\square$

Remark 4.19. In [20], Simpson proves an analogous result for strict n-categories with weak inverses (see Theorem 2.1.C of op. cit.). See also Proposition 1.7 of [4] for the case of strict ∞-groupoids.

5. FUNDAMENTAL ∞-GROUPOID FUNCTORS

5.1. Formal coherators. A *formal coherator* is a weakly initial object C of the category of contractible globular extensions, such that any globular functor $\Theta_0 \to C$ is bijective on objects. Recall that an object X of a category $\mathcal{C}$ is said to be *weakly initial* if for any object Y of $\mathcal{C}$ there exists at least one arrow from X to Y. Proposition 2.14 shows that every coherator is a formal coherator.

In the rest of this section, we fix a formal coherator C.

5.2. The fundamental ∞-groupoid functor of a contractible globular extension. Let $\mathcal{M}$ be a contractible globular extension whose underlying category is cocomplete. By definition of C, there exists a globular functor $F : C \to \mathcal{M}$. Since $\mathcal{M}$ is cocomplete, this functor induces an adjunction

$$R_{C,F} : \widehat{C} \to \mathcal{M}, \qquad \Pi_{C,F} : \mathcal{M} \to \widehat{C},$$

where $R_{C,F}$ is the unique extension of F to $\widehat{C}$ preserving colimits and $\Pi_{C,F}$ is given by the following formula: for every object X of $\mathcal{M}$, $\Pi_{C,F}(X) = (S \mapsto \mathsf{Hom}_{\mathcal{M}}(F(S), X))$. Since the functor F is globular, the presheaf $\Pi_{C,F}(X)$ is globular and the previous adjunction induces an adjunction

$$R_{C,F} : \infty\text{-}\mathcal{G}\mathrm{pd}_C \to \mathcal{M}, \qquad \Pi_{C,F} : \mathcal{M} \to \infty\text{-}\mathcal{G}\mathrm{pd}_C,$$

which we will denote the same way. If X is an object of $\mathcal{M}$, we will call $\Pi_{C,F}(X)$ the *fundamental ∞-groupoid* of X. Note that it depends a priori on F. However, the underlying globular set of $\Pi_{C,F}$ depends neither on F nor on C: its set of n-arrows is given by

$$\Pi_{C,F}(X)_n = \mathsf{Hom}_{\mathcal{M}}(\mathrm{D}_n, X).$$

Examples 5.3. Let $\mathcal{M} = \mathcal{T}\!op$. We have seen in Example 2.7.1 that the globular extension $\mathcal{T}\!op$ is contractible. By the previous paragraph we get an adjunction

$$R_{C,F} : \infty\text{-}\mathcal{G}\mathrm{pd}_C \to \mathcal{T}\!op, \qquad \Pi_{C,F} : \mathcal{T}\!op \to \infty\text{-}\mathcal{G}\mathrm{pd}_C,$$

which depends on a globular functor $F : C \to \mathcal{T}\!op$. If X is a topological space, we have

$$\Pi_{C,F}(X)_n = \mathsf{Hom}_{\mathcal{T}\!op}(\mathbf{D}_n, X).$$

In particular,

- $\Pi_{C,F}(X)_0$ is the set of points of X;
- $\Pi_{C,F}(X)_1$ is the set of paths of X;
- $\Pi_{C,F}(X)_2$ is the set of (relative) homotopies between paths of X;
- $\Pi_{C,F}(X)_3$ is the set of (relative) homotopies between (relative) homotopies between paths of X;
- etc.

5.4. If D is a globular extension, then the Yoneda functor $D \to \widehat{D}$ factors through the category $\mathrm{Mod}(D)$ and we get a functor $D \to \mathrm{Mod}(D)$. In particular, if D is a contractible globular extension, we get a functor

$$y_D : D \to \infty\text{-}\mathcal{G}\mathrm{pd}_D.$$

Proposition 5.5. *Let $\mathcal{M}$ be a cocomplete contractible globular extension and let $F : C \to \mathcal{M}$ be a globular functor. Then for any object X of $\mathcal{M}$ and any $n \geq 1$, the groupoid $\varpi_n(\Pi_{C,F}(X))$ is equal to $\varpi_n(y_{\mathcal{M}}(X))$. In particular, $\varpi_n(\Pi_{C,F}(X))$ depends neither on F nor on C.*

Proof. Consider the following diagram:

$$
\begin{array}{ccc}
 & \mathcal{M} & \\
y_{\mathcal{M}} \swarrow & & \searrow \Pi_{C,F} \\
\infty\text{-}\mathcal{G}\mathrm{pd}_{\mathcal{M}} & \xrightarrow{\ F^*\ } & \infty\text{-}\mathcal{G}\mathrm{pd}_C \\
\varpi_n \searrow & & \swarrow \varpi_n \\
 & \mathcal{G}\mathrm{pd} &
\end{array}
\quad .
$$

The upper triangle is commutative by definition of $\Pi_{C,F}$ and the lower triangle is commutative by Proposition 5.5. Hence the result. $\qquad\square$

Remark 5.6. Let $F, F' : C \to \mathcal{M}$ be two globular functors. Since $\Pi_{C,F}(X)$ and $\Pi_{C,F'}(X)$ have the same underlying globular set, we can consider the identity morphism (of globular sets) from $\Pi_{C,F}(X)$ to $\Pi_{C,F'}(X)$. This morphism is *not* a morphism of ∞-groupoids. Our feeling is that it should be part of the data defining a *weak* morphism of ∞-groupoids (whatever that might mean). The previous proposition would then show that this weak morphism is a weak equivalence and so that $\Pi_{C,F}$ does not depend on F in some homotopy category.

5.7. In the rest of this section, we will explain how to construct a fundamental ∞-groupoid functor $\mathcal{M} \to \infty\text{-}\mathcal{G}\mathrm{pd}_C$ when $\mathcal{M}$ is a model category in which every object is fibrant.

We will denote by $\varnothing$ (resp. $*$) the initial (resp. terminal) object of a category. If X is an object of a model category $\mathcal{M}$, we will say that X is *weakly contractible* if the unique morphism $X \to *$ is a weak equivalence.

5.8. Cofibrant and weakly contractible functors (definition). Let $\mathcal{M}$ be a model category endowed with a functor $F : \mathbb{G} \to \mathcal{M}$. For $n \geq 0$, we define an object S^{n-1} endowed with a map $i_n : \mathrm{S}^{n-1} \to \mathrm{D}_n$ in the following way. For $n = 0$, we set

$$\mathrm{S}^{-1} = \varnothing$$

and we define

$$i_0 : \mathrm{S}^{-1} \to \mathrm{D}_0$$

as the unique morphism from the initial object to D_0. For $n \geq 1$, we set

$$\mathrm{S}^{n-1} = (\mathrm{D}_{n-1}, i_{n-1}) \amalg_{\mathrm{S}^{n-2}} (i_{n-1}, \mathrm{D}_{n-1})$$

and

$$i_n = (\tau_n, \sigma_n) : \mathrm{S}^{n-1} \to \mathrm{D}_n.$$

Let us justify that i_n is well-defined. We have to check that $\sigma_n i_{n-1} = \tau_n i_{n-1}$. But by looking at the two components of this equality, one sees that it is equivalent to the coglobular relations.

We will say that the functor $F : \mathbb{G} \to \mathcal{M}$ is *cofibrant* if for every $n \geq 0$, the morphism $i_n : \mathrm{S}^{n-1} \to \mathrm{D}_n$ is a cofibration in $\mathcal{M}$. We will say that the functor $F : \mathbb{G} \to \mathcal{M}$ is *weakly contractible* if for any $n \geq 0$, the object D_n is weakly contractible in $\mathcal{M}$.

Remark 5.9. If $\mathcal{M}$ is a model category, the category $\underline{\mathsf{Hom}}(\mathbb{G}, \mathcal{M})$ of functors from $\mathbb{G}$ to $\mathcal{M}$ is endowed with a model category structure coming from the fact that $\mathbb{G}$ is a direct category: the so-called Reedy model structure. It is easy to show that a functor $F : \mathbb{G} \to \mathcal{M}$ is cofibrant in this model category if and only if it is a cofibration in the sense of the previous paragraph. Indeed, the n-th latching object of a functor $F : \mathbb{G} \to \mathcal{M}$ is exactly S^{n-1}. Moreover, a functor $F : \mathbb{G} \to \mathcal{M}$ is weakly contractible in this model category structure if and only if it is weakly contractible in the sense of the previous paragraph. In particular, a cofibrant and weakly contractible functor $F : \mathbb{G} \to \mathcal{M}$ is nothing but a cofibrant replacement of the terminal object of $\underline{\mathsf{Hom}}(\mathbb{G}, \mathcal{M})$. Such a functor hence exists and is in some sense unique.

We will make no use of this observation in the rest of the article. In particular, we will prove by hand the existence of a cofibrant weakly contractible functor (see paragraph 5.13). We refer the reader to Chapter 15 of [12] for the theory of Reedy model structures.

Example 5.10. Let $R : \mathbb{G} \to \mathcal{T}\!op$ be the functor defining the globular extension structure of $\mathcal{T}\!op$. In this case, the morphism i_n is nothing but the inclusion of the $(n-1)$-sphere $\mathbf{S}^{n-1}$ as the boundary of the n-disk $\mathbf{D}_n$ and is hence a cofibration. On the other hand, the disks are of course contractible. The functor R is thus cofibrant and weakly contractible.

Proposition 5.11. *Let $\mathcal{M}$ be a model category endowed with a cofibrant and weakly contractible functor $F : \mathbb{G} \to \mathcal{M}$. Then every globular sum in $\mathcal{M}$ is weakly contractible.*

Proof. Let us first prove that the σ_n's and the τ_n's are cofibrations. Let $n \geq 1$. We have $\sigma_n = i_n \varepsilon_2$ and $\tau_n = i_n \varepsilon_1$, where $\varepsilon_1, \varepsilon_2 : \mathrm{D}_{n-1} \to \mathrm{S}^{n-1}$ are the two canonical morphisms. The morphism i_n is a cofibration by hypothesis. Moreover, the ε_i's are both pushouts of i_{n-1} and hence are cofibrations. It follows that σ_n and τ_n are cofibrations.

Let us now prove the assertion. We need to show that for every table of dimensions, the globular sum associated to the table is weakly contractible. We prove this by induction on the width n of the table. If $n = 1$, the globular sum is a D_n which is weakly contractible by assumption. Otherwise, let

$$S = \begin{pmatrix} i_1 & & \cdots & & i_n \\ & i_1' & \cdots & i_{n-1}' & \end{pmatrix}$$

be our table of dimensions and X be the globular sum associated to S. Set

$$T = \begin{pmatrix} i_1 & & \cdots & & i_{n-1} \\ & i_1' & \cdots & i_{n-2}' & \end{pmatrix}$$

and denote by Y the globular sum associated to T. We have

$$X = Y \amalg_{\mathrm{D}_{i_{n-1}'}} \mathrm{D}_{i_n}.$$

More precisely, the commutative square

$$
\begin{array}{ccc}
\mathrm{D}_{i'_{n-1}} & \xrightarrow{\ \tau^{i_n}_{i'_{n-1}}\ } & \mathrm{D}_{i_n} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
$$

is cocartesian. But the top horizontal arrow is a cofibration between weakly contractible objects and is thus a trivial cofibration. Hence the canonical morphism $Y \to X$ is a trivial cofibration. Since by induction Y is weakly contractible, X is also weakly contractible. Hence the result. $\qquad\square$

Proposition 5.12. *Let $\mathcal{M}$ be a model category endowed with a cofibrant and weakly contractible functor $F : \mathbb{G} \to \mathcal{M}$. Assume that every object of $\mathcal{M}$ is fibrant. Then the globular extension $(\mathcal{M}, F)$ is contractible.*

Proof. The data of an admissible pair $(f, g) : \mathrm{D}_n \to X$ is equivalent to the data of a morphism $\mathrm{S}^n \to X$. Indeed, every morphism $k : \mathrm{S}^n \to X$ can be written $k = (f, g)$, where $f, g : \mathrm{D}_n \to X$ satisfy $f i_n = g i_n$. But the two components of this equality are precisely the relations showing that f and g are globularly parallel.

Moreover, a morphism $h : \mathrm{D}_n \to X$ is a lifting of the admissible pair $(f, g) : \mathrm{D}_n \to X$ if and only if the triangle

$$
\begin{array}{ccc}
& & \mathrm{D}_{n+1} \\
& \nearrow \hspace{-0.5em} {}^{h} & \\
{}_{i_{n+1}}\!\uparrow & & \searrow \\
\mathrm{S}^n & \xrightarrow[(f,g)]{} & X
\end{array}
$$

commutes. Indeed, the commutativity of the triangle is equivalent to the equalities $h i_n \varepsilon_2 = f$ and $h i_n \varepsilon_1 = g$. But we have $i_n \varepsilon_2 = \sigma_n$ and $i_n \varepsilon_1 = \tau_n$.

Consider now the commutative square

$$
\begin{array}{ccc}
\mathrm{S}^n & \xrightarrow{\ (f,g)\ } & X \\
{}_{i_{n+1}}\!\downarrow & & \downarrow \\
\mathrm{D}_{n+1} & \longrightarrow & *
\end{array}.
$$

By hypothesis, the morphism i_{n+1} is a cofibration. Moreover, by the previous proposition, the globular sum X is weakly contractible. Since by assumption, every object of $\mathcal{M}$ is fibrant, the morphism $X \to *$ is a trivial fibration. Hence there exists a lifting $\mathrm{D}_{n+1} \to X$. This shows that the globular extension $(\mathcal{M}, F)$ is contractible. $\qquad\square$

5.13. Cofibrant and weakly contractible functors (construction). Let $\mathcal{M}$ be a model category. We will now construct a cofibrant and weakly contractible functor from $\mathbb{G}$ to $\mathcal{M}$.

We define by induction on $n \geq 0$ an object D_n of $\mathcal{M}$ and a cofibration $i_n : \mathrm{S}^{n-1} \to \mathrm{D}_n$, where

$$
\mathrm{S}^{-1} = \varnothing \quad \text{and} \quad \mathrm{S}^{n-1} = (\mathrm{D}_{n-1}, i_{n-1}) \amalg_{\mathrm{S}^{n-2}} (i_{n-1}, \mathrm{D}_{n-1}), \quad n \geq 1,
$$

in the following way. For $n = 0$, we define D_0 as a cofibrant replacement of the terminal object and $i_0 : \mathrm{S}^{-1} \to \mathrm{D}_0$ as the unique morphism from the initial object to D_0. For $n \geq 1$, consider the morphism

$$(1_{\mathrm{D}_{n-1}}, 1_{\mathrm{D}_{n-1}}) : \mathrm{S}^{n-1} \to \mathrm{D}_{n-1}$$

and factor it as a cofibration i_n, followed by a weak equivalence p_n. We define D_n as the middle object of this factorization

$$\mathrm{S}^{n-1} \xrightarrow{i_n} \mathrm{D}_n \xrightarrow{p_n} \mathrm{D}_{n-1}.$$

Note that since p_n is a weak equivalence and D_0 is weakly contractible, by induction on $n \geq 0$, the object D_n is weakly contractible.

We now set

$$\sigma_n = i_n \varepsilon_2 \quad \text{and} \quad \tau_n = i_n \varepsilon_1, \quad n \geq 1,$$

where $\varepsilon_1, \varepsilon_2 : \mathrm{D}_{n-1} \to \mathrm{S}^{n-1}$ are the canonical morphisms. Let us prove the coglobular relations. Let $n \geq 1$. By definition of S^n, we have $\varepsilon'_1 i_n = \varepsilon'_2 i_n$, where $\varepsilon'_1, \varepsilon'_2 : \mathrm{D}_n \to \mathrm{S}^n$ are the canonical morphisms. We thus have

$$\sigma_{n+1}\sigma_n = i_{n+1}\varepsilon'_2 i_n \varepsilon_2 = i_{n+1}\varepsilon'_1 i_n \varepsilon_2 = \tau_{n+1}\sigma_n$$

and

$$\sigma_{n+1}\tau_n = i_{n+1}\varepsilon'_2 i_n \varepsilon_1 = i_{n+1}\varepsilon'_1 i_n \varepsilon_1 = \tau_{n+1}\tau_n.$$

We have thus defined a functor $\mathbb{G} \to \mathcal{M}$. This functor is cofibrant and weakly contractible by construction.

5.14. The fundamental ∞-groupoid functor of a model category. Let now $\mathcal{M}$ be a model category in which every object is fibrant. By the previous paragraph, there exists a cofibrant and weakly contractible functor $F : \mathbb{G} \to \mathcal{M}$. By Proposition 5.12, the globular extension $(\mathcal{M}, F)$ is contractible. We can thus apply paragraph 5.2 and we get an adjunction

$$R_{C,K} : \infty\text{-}\mathcal{G}\mathrm{pd}_C \to \mathcal{M}, \qquad \Pi_{C,K} : \mathcal{M} \to \infty\text{-}\mathcal{G}\mathrm{pd}_C,$$

which depends on a globular functor $K : C \to \mathcal{M}$.

If X is an object of $\mathcal{M}$, we will call $\Pi_{C,K}(X)$ the *fundamental ∞-groupoid* of X. Note that it depends a priori on F and K. In particular, its underlying globular set depends on F. However, this ∞-groupoid should not depend on F and K in some weak sense (see Remark 5.6). We will show in the next section that the homotopy groups of this ∞-groupoid are independent of F and K.

Remark 5.15. If $\mathcal{M}$ is a combinatorial model category, we can also construct a fundamental ∞-groupoid functor. Indeed, by a theorem of Dugger ([9, Theorem 1.1]), such an $\mathcal{M}$ is Quillen equivalent to some left Bousfield localization $\mathcal{N}$ of a category of simplicial presheaves endowed with the projective model structure. The model category $\mathcal{N}$ is combinatorial and every cofibration of $\mathcal{N}$ is a monomorphism. By a theorem of Nikolaus ([18, Corollary 2.21]), such a model category is Quillen equivalent to a model category $\mathcal{P}$ in which every object is fibrant. Although these two Quillen equivalences do not compose, we obtain a functor $Q : \mathcal{M} \to \mathcal{P}$ "inducing" an equivalence on homotopy categories. We can thus define a fundamental ∞-groupoid functor as the composition

$$\Pi_\infty : \mathcal{M} \to \mathcal{P} \xrightarrow{\Pi_\infty} \infty\text{-}\mathcal{G}\mathrm{pd}_C,$$

where $\Pi_\infty : \mathcal{P} \to \infty\text{-}\mathcal{G}\mathrm{pd}_C$ is a fundamental ∞-groupoid functor in the sense of the previous paragraph.

Remark 5.16. We do not need the full strength of a model category structure to construct the fundamental ∞-groupoid functor. Recall that if $\mathcal{C}$ is category, a *weak factorization system* consists of two classes of morphisms L and R of $\mathcal{C}$ satisfying the following properties:

(1) every morphism f of $\mathcal{C}$ factors as $f = pi$ with i in L and p in R;
(2) L is the class of morphisms of $\mathcal{C}$ having the left lifting property with respect to R;
(3) R is the class of morphisms of $\mathcal{C}$ having the right lifting property with respect to L.

Let $\mathcal{M}$ be a cocomplete category admitting a terminal object, endowed with a weak factorization system (L, R). We will think of the elements of L as cofibrations and of the elements of R as trivial fibrations (and in particular as weak equivalences). We can construct as in paragraph 5.13 a functor $F : \mathbb{G} \to \mathcal{M}$ by replacing every factorization as a cofibration followed by a weak equivalence, by a factorization as a morphism of L followed by a morphism of R. It is not true in general that $(\mathcal{M}, F)$ is a contractible globular extension: we need to add a hypothesis saying that in some sense every object of $\mathcal{M}$ is fibrant.

We will say that an object X of $\mathcal{M}$ is *fibrant and weakly contractible* if the unique morphism $X \to *$ is in R. Suppose now that our weak factorization system satisfies the following additional property: for every cocartesian square

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X_2 \\
\downarrow & & \downarrow \\
X_3 & \longrightarrow & X_4
\end{array}
$$

in $\mathcal{M}$, if X_1, X_2, X_3 are fibrant and weakly contractible, then so is X_4. Note that this hypothesis is satisfied when we consider the factorization system of cofibrations and trivial fibrations of a model category in which every object is fibrant.

Under this hypothesis, the proofs of Propositions 5.11 and 5.12 can easily be adapted and we obtain that $(\mathcal{M}, F)$ is a contractible globular extension. We can thus define a fundamental ∞-groupoid functor $\Pi_\infty : \mathcal{M} \to \infty\text{-}\mathcal{G}\mathrm{pd}_C$.

6. Quillen's theory of π_1 in a model category

The purpose of this section is to recall to the reader some definitions and facts about Quillen's theory of π_1 in a model category which was introduced in Section I.2 of [19].

6.1. Notation and terminology about model categories. If $\mathcal{M}$ is a model category and X, Y are two objects of $\mathcal{M}$, we will denote by $[X, Y]_{\mathcal{M}}$ the set of morphisms between X and Y in the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$. If $\mathcal{M}$ is understood, we will simply denote this set by $[X, Y]$. Recall that when X is cofibrant and Y is fibrant, $[X, Y]$ is in canonical bijection with the set of morphisms $X \to Y$ up to (left or right) homotopy.

Let $\mathcal{M}$ be a model category. The definition of cylinder objects varies from author to author. We will use Quillen's original definition: a *cylinder object* of an object A of $\mathcal{M}$

is an object C of $\mathcal{M}$ endowed with a factorization

$$A \amalg A \xrightarrow{(i_1, i_0)} C \xrightarrow{s} A$$

of the codiagonal of A as a cofibration followed by a weak equivalence. Dually, a *path object* of an object B of $\mathcal{M}$ is an object P of $\mathcal{M}$ endowed with a factorization

$$B \xrightarrow{r} P \xrightarrow{(p_1, p_0)} B \times B$$

of the diagonal of B as a weak equivalence followed by a fibration. We will always use the letters i, s, r and p to denote the structural maps of cylinder and path objects.

If $f, g : A \to B$ are two morphisms of $\mathcal{M}$, a *left homotopy* from f to g is a morphism $h : C \to B$, where C is a cylinder object of A, such that $hi_0 = f$ and $hi_1 = g$. Note that we have inverted Quillen's original direction for homotopies (because we have exchanged i_0 and i_1) in order to be coherent with our convention for globular sums. We apply the same treatment to right homotopies.

We will use the following easy fact several times.

6.2. Existence and uniqueness of morphisms of path objects. Let $f : A \to A'$ be a morphism in a model category $\mathcal{M}$. Let P (resp. P') be a path object of A (resp. of A'). Suppose moreover that $r : A \to P$ is a cofibration. Then there exists a morphism g such that the square

$$
\begin{array}{ccccc}
A & \xrightarrow{r} & P & \xrightarrow{(p_1, p_0)} & A \times A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f \times f} \\
A' & \xrightarrow{r'} & P' & \xrightarrow{(p'_1, p'_0)} & A' \times A'
\end{array}
$$

is commutative. Indeed, any lifting of the commutative square

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & A' & \xrightarrow{r'} & P' \\
\downarrow{\scriptstyle r} & & & & \downarrow{\scriptstyle (p'_1, p'_0)} \\
P & \xrightarrow{(p_1, p_0)} & A \times A & \xrightarrow{f \times f} & A' \times A'
\end{array}
$$

gives such a morphism g. Moreover, such a morphism is a weak equivalence and is unique up to homotopy in the category of objects of $\mathcal{M}$ under A and over $A' \times A'$ (see Proposition 7.6.14 of [12]).

From now on, we fix a model category $\mathcal{M}$, a cofibrant object A of $\mathcal{M}$, a fibrant object B of $\mathcal{M}$ and two morphisms $f, g : A \to B$.

6.3. 2-homotopies and correspondences. Let C and C' be two cylinder objects of A. A 2-*cylinder object* of C and C' is an object D of $\mathcal{M}$ endowed with a factorization

$$(C', (i'_1, i'_0)) \amalg_{A \amalg A} (C, (i_1, i_0)) \xrightarrow{(j_1, j_0)} D \xrightarrow{t} A$$

of the morphism

$$(s', s) : (C', (i'_1, i'_0)) \amalg_{A \amalg A} (C, (i_1, i_0)) \to A$$

as a cofibration followed by a weak equivalence. If $C = C'$, we will simply say that D is a 2-cylinder object of C.

If $h : C \to B$ and $h' : C' \to B$ are two left homotopies from f to g, then a *left 2-homotopy* from h to h' is a morphism $H : D \to B$ such that $Hj_0 = h$ and $Hj_1 = h'$. If such an H exists, we will say that h and h' are *left* 2-*homotopic*.

Let C be a cylinder object of A and let P be a path object of B. If $h : C \to B$ is a left homotopy from f to g and $k : A \to P$ is a right homotopy from f to g, a *correspondence* between h and k is a map $H : C \to P$ such that

$$p_0 H = h, \quad Hi_0 = k, \quad p_1 H = gs \quad \text{and} \quad Hi_1 = rg.$$

If such a correspondence exists, we will say that h and k *correspond*.

Lemma 6.4 (Quillen)**.**

(1) *Let $h : C \to B$ be a left homotopy from f to g. Then for every path object P of B, there exists a right homotopy $k : A \to P$ corresponding to h.*

(2) *Let $h : C \to B$ and $h' : C' \to B$ be two left homotopies from f to g. If h corresponds to a right homotopy k, then h is left* 2-*homotopic to h' if and only if h' corresponds to k. More precisely, if D is a fixed 2-cylinder object of C and C', then there exists a left* 2-*homotopy $D \to B$ from h to h' if and only if h' and k correspond.*

Proof.

(1) See Lemma 1 of [19, Section I.2].

(2) See Lemma 2 of [19, Section I.2]. The fact that one can fix the 2-cylinder D is not stated but appears clearly in the proof.

$\square$

6.5. The set $\pi_1(A, B; f, g)$**.** We will denote by $\pi_1(A, B; f, g)$ the class of left homotopies of $\mathcal{M}$ from f to g, up to left 2-homotopy. It is not clear a priori that $\pi_1(A, B; f, g)$ is a set. But by the previous lemma (and its dual), $\pi_1(A, B; f, g)$ is in canonical bijection with the class of left homotopies from a fixed cylinder C up to left 2-homotopy. This is obviously a set. Note that we can even ask for the left 2-homotopies to use a fixed 2-cylinder of C.

Dually, we can consider right homotopies up to right 2-homotopies in an appropriate sense. The previous lemma shows that we get this way a set in canonical bijection with $\pi_1(A, B; f, g)$.

6.6. The groupoid $\Pi_1(A, B)$**.** Let $\Pi_1(A, B)$ be the graph defined in the following way. The objects of $\Pi_1(A, B)$ are the morphisms from A to B of $\mathcal{M}$. If $f, g : A \to B$ are two such objects, the set of arrows from f to g in $\Pi_1(A, B)$ is $\pi_1(A, B; f, g)$.

Let $f_1, f_2, f_3 : A \to B$ be three morphisms of $\mathcal{M}$, and let $h : C \to B$ and $h' : C' \to B$ be two left homotopies respectively from f_1 to f_2 and from f_2 to f_3. Then

$$(C', i'_0) \amalg_A (i_1, C)$$

is a left cylinder for the factorization

$$A \amalg A \xrightarrow{i_1 \amalg i_0} (C', i'_0) \amalg_A (i_1, C) \xrightarrow{(s', s)} A \,,$$

and

$$(C', i'_0) \amalg_A (i_1, C) \xrightarrow{(h', h)} B$$

is a left homotopy from f_1 to f_3.

Proposition 6.7 (Quillen)**.** *The above construction defines a map*

$$\pi_1(A, B; f_2, f_3) \times \pi_1(A, B; f_1, f_2) \to \pi_1(A, B; f_1, f_3)$$

for every $f_1, f_2, f_3 : A \to B$ and these maps induce a groupoid structure on $\Pi_1(A, B)$.

Proof. See Proposition 1 of [19, Section I.2]. $\square$

6.8. Under and over model categories. If X is an object of $\mathcal{M}$, we will denote by $X \backslash \mathcal{M}$ the category of objects of $\mathcal{M}$ under X, that is, the category whose objects are pairs (Y, u) where Y is an object of $\mathcal{M}$ and $u : X \to Y$ is a morphism of $\mathcal{M}$, and whose morphisms from (Y, u) to (Y', u') are morphisms $v : Y \to Y'$ of $\mathcal{M}$ such that $uv = u'$. This category inherits a structure of model category from the one on $\mathcal{M}$: a morphism of $X \backslash \mathcal{M}$ is a cofibration (resp. a fibration, resp. a weak equivalence) if the underlying morphism of $\mathcal{M}$ is a cofibration (resp. a fibration, resp. a weak equivalence). Note that an object (Y, u) is cofibrant (resp. fibrant) in $X \backslash \mathcal{M}$ if and only if u is a cofibration of $\mathcal{M}$ (resp. if and only if Y is a fibrant object of $\mathcal{M}$).

Dually, if X is an object of $\mathcal{M}$, we will denote by $\mathcal{M}/X$ the category of objects of $\mathcal{M}$ over X. This category is nothing but $(X \backslash M^{\mathrm{o}})^{\mathrm{o}}$, and all the above statements can be dualized.

In what follows, we will often work in the category $(A \amalg A) \backslash \mathcal{M}$. The morphisms $f, g : A \to B$ induce a morphism $(g, f) : A \amalg A \to B$ and we get an object $(B, (g, f))$ of $(A \amalg A) \backslash \mathcal{M}$. This object is fibrant since B is fibrant in $\mathcal{M}$. Similarly, from a cylinder object C of A, we get an object $(C, (i_1, i_0))$ of $(A \amalg A) \backslash \mathcal{M}$. This object is cofibrant since (i_1, i_0) is a cofibration of $\mathcal{M}$. Each time we will consider B and C as objects of $(A \amalg A) \backslash \mathcal{M}$, they will be endowed with the morphisms we have just defined.

Proposition 6.9. *Let C be a cylinder object of A. A morphism $h : C \to B$ of $\mathcal{M}$ defines a morphism $(C, (i_1, i_0)) \to (B, (g, f))$ in $(A \amalg A) \backslash \mathcal{M}$ if and only if h is a homotopy from f to g in $\mathcal{M}$. This correspondence induces a bijection*

$$\pi_1(A, B; f, g) \cong [(C, (i_1, i_0)), (B, (g, f))]_{(A \amalg A) \backslash \mathcal{M}}.$$

Proof. The first assertion is obvious. Let us prove the second one. Since the objects $(C, (i_1, i_0))$ and $(B, (g, f))$ are respectively cofibrant and fibrant,

$$[(C, (i_1, i_0)), (B, (g, f))]_{(A \amalg A) \backslash \mathcal{M}}$$

is the set of morphisms $(C, (i_1, i_0)) \to (B, (g, f))$ up to left homotopy in $(A \amalg A) \backslash \mathcal{M}$. Moreover, by paragraph 6.5, $\pi_1(A, B; f, g)$ is the set of left homotopies $C \to B$ between f and g up to left 2-homotopy in $\mathcal{M}$.

Let $h, h' : (C, (i_1, i_0)) \to (B, (g, f))$ be two morphisms. We have to show that h and h' are left homotopic in $(A \amalg A) \backslash \mathcal{M}$ if and only if they are 2-homotopic as homotopies between f and g in $\mathcal{M}$.

Consider the morphism $(1_C, 1_C) : C \amalg_{A \amalg A} C \to C$ of $\mathcal{M}$. Let

$$C \amalg_{A \amalg A} C \xrightarrow{k} D \xrightarrow{q} C$$

be a factorization of this morphism as a cofibration followed by a weak equivalence. We get an object (D, k) under $A \amalg A$ and the above factorization makes (D, k) a cylinder object of $(C, (i_1, i_0))$ in $(A \amalg A) \backslash \mathcal{M}$. The morphisms h and h' are left homotopic in $(A \amalg A) \backslash \mathcal{M}$ if and only if there exists a left homotopy between them in $(A \amalg A) \backslash \mathcal{M}$ using the cylinder (D, k).

Using the weak equivalence $s : C \to A$ of $\mathcal{M}$, we obtain a factorization

$$C \amalg_{A \amalg A} C \xrightarrow{k} D \xrightarrow{sq} A$$

in $\mathcal{M}$, making D a 2-cylinder of C in $\mathcal{M}$. The left homotopies h and h' are left 2-homotopic if and only if there exists a left 2-homotopy between them in $\mathcal{M}$ using the 2-cylinder D (see paragraph 6.5).

But it is obvious that a morphism $H : D \to B$ of $\mathcal{M}$ induces a left homotopy between h and h' in $(A \amalg A) \backslash \mathcal{M}$ if and only if H is a left 2-homotopy between h and h' in $\mathcal{M}$. Hence the result. $\square$

Remark 6.10. Let C be a cylinder object of A and let P be a path object of B. The previous proposition and its dual proposition imply that there is a canonical bijection

$$[(C, (i_1, i_0)), (B, (g, f))]_{(A \amalg A) \backslash \mathcal{M}} \cong [(A, (g, f)), (P, (p_1, p_0))]_{\mathcal{M}/(B \times B)}.$$

Explicitly, a left homotopy $h : C \to B$ from f to g is sent to a right homotopy $k : A \to P$ corresponding to h.

Lemma 6.11. *Let B' be a second fibrant object of $\mathcal{M}$ and let $v : B \to B'$ be a morphism of $\mathcal{M}$. Let C be a cylinder object of A and let P (resp. P') be a path object of B (resp. of B'). Assume that $r : B \to P$ is a cofibration. Let $w : P \to P'$ be any morphism making the diagram*

$$
\begin{array}{ccccc}
B & \xrightarrow{\ r\ } & P & \xrightarrow{(p_1, p_0)} & B \times B \\
{\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v \times v} \\
B' & \xrightarrow[\ r'\ ]{} & P' & \xrightarrow[(p'_1, p'_0)]{} & B' \times B'
\end{array}
$$

commute (such a morphism exists by paragraph 6.2). Then the induced map

$$w \circ - : [(A, (g, f)), (P, (p_1, p_0))]_{\mathcal{M}/(B \times B)} \to [(A, (vg, vf)), (P', (p'_1, p'_0))]_{\mathcal{M}/(B' \times B')}$$

does not depend on the choice of w, and the square

$$
\begin{array}{ccc}
[(C, (i_1, i_0)), (B, (g, f))]_{(A \amalg A) \backslash \mathcal{M}} & \xrightarrow{\ \sim\ } & [(A, (g, f)), (P, (p_1, p_0))]_{\mathcal{M}/(B \times B)} \\
{\scriptstyle v \circ -}\downarrow & & \downarrow{\scriptstyle w \circ -} \\
[(C, (i_1, i_0)), (B', (vg, vf))]_{(A \amalg A) \backslash \mathcal{M}} & \xrightarrow[\ \sim\ ]{} & [(A, (vg, vf)), (P', (p'_1, p'_0))]_{\mathcal{M}/(B' \times B')},
\end{array}
$$

where the horizontal arrows are the bijections of the previous remark, is commutative.

Proof. It suffices to show that the square of the statement is commutative. Let $h : C \to B$ be a left homotopy from f to g and let $k : A \to P$ be a right homotopy corresponding to h via a correspondence $H : C \to P$. We have to show that the left homotopy $vh : C \to B'$ from vf to vg corresponds to the right homotopy $wk : A \to P'$. It is immediate that $wH : C \to P'$ is the desired correspondence. $\square$

6.12. Functoriality of Π_1**.** Let B' be a second fibrant object of $\mathcal{M}$ and let $v : B \to B'$ be a morphism of $\mathcal{M}$. The morphism v induces a morphism of graphs

$$\Pi_1(A, v) : \Pi_1(A, B) \to \Pi_1(A, B')$$

by sending a morphism $f : A \to B$ of $\mathcal{M}$ to $vf : A \to B'$, and a left homotopy $h : C \to B$ representing an element of $\pi_1(A, B; f, g)$ to the left homotopy $vh : C \to B'$. It is easy to see that this morphism of graphs is a functor. Note that in the bijection of Proposition 6.9, the mapping $h \mapsto vh$ corresponds to the post-composition by v seen as a morphism of $\mathrm{Ho}((A \amalg A)\backslash\mathcal{M})$.

Dually, if A' is a second cofibrant object and $u : A' \to A$ is a morphism of $\mathcal{M}$, then u induces a functor

$$\Pi_1(u, B) : \Pi_1(A, B) \to \Pi_1(A', B).$$

Note that we need to consider *right* homotopies to define this functor.

By Lemma 3 of [19, Section I.2], the square

$$
\begin{array}{ccc}
\Pi_1(A, B) & \longrightarrow & \Pi_1(A', B) \\
\downarrow & & \downarrow \\
\Pi_1(A, B') & \longrightarrow & \Pi_1(A', B')
\end{array}
$$

is commutative. This also follows easily from Lemma 6.11. We can thus define a functor

$$\Pi_1(u, v) : \Pi_1(A, B) \to \Pi_1(A', B').$$

Proposition 6.13. *Let $u : A' \to A$ be a weak equivalence between cofibrant objects and let $v : B \to B'$ be a weak equivalence between fibrant objects. Then the functor*

$$\Pi_1(u, v) : \Pi_1(A, B) \to \Pi_1(A', B')$$

is an equivalence of categories.

Proof. By definition of $\Pi_1(u, v)$ and by duality, we can assume that u is an identity. But in the bijection of Proposition 6.9, $\Pi_1(A, v)$ is the post-composition by v in the homotopy category of $(A \amalg A)\backslash\mathcal{M}$. Since v is a weak equivalence, this post-composition is a bijection, hence the result. $\square$

7. Quillen homotopy groups in a model category

In this section, we fix a model category $\mathcal{M}$ in which every object is fibrant.

7.1. Connected components. Let X be an object of $\mathcal{M}$. The set $\pi_0(X)$ of *connected components* of X is defined by

$$\pi_0(X) = [*, X].$$

This set can be described in terms of homotopy classes thanks to the formula

$$\pi_0(X) \cong [\mathrm{D}_0, X],$$

where D_0 is any cofibrant contractible object of $\mathcal{M}$. It is obvious that π_0 defines a functor from $\mathcal{M}$ to the category of sets and that this functor sends weak equivalences to bijections.

7.2. Based objects. Let X be an object of $\mathcal{M}$. A *base point* x of X is a morphism $D_0 \to X$ of $\mathcal{M}$, where D_0 is cofibrant and weakly contractible. We will say that an object of $\mathcal{M}$ endowed with a base point is a *based object* of $\mathcal{M}$.

Let $(X, x : D_0 \to X)$ and $(X', x' : D'_0 \to X)$ be two based objects of $\mathcal{M}$. A *morphism of based objects* from (X, x) to (X', x') is given by morphisms $f : X \to X'$ and $f_0 : D_0 \to D'_0$ of $\mathcal{M}$ making the square

$$
\begin{array}{ccc}
D_0 & \xrightarrow{\ x\ } & X \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\
D'_0 & \xrightarrow{\ x'\ } & X'
\end{array}
$$

commute. Note that such an f_0 is necessarily a homotopy equivalence since D_0 and D'_0 are both cofibrant, fibrant and weakly contractible. We will denote such a morphism of based objects by (f, f_0).

We adopt the following convention on notation: if (X, x) (resp. (X', x')) is a based object of $\mathcal{M}$, then the source of x (resp. of x') will be denoted by D_0 (resp. by D'_0), unless otherwise specified.

7.3. The fundamental group. Let $(X, x : D_0 \to X)$ be a based object of $\mathcal{M}$. The *fundamental group* of (X, x) is the group $\pi_1(D_0, X; x, x)$. We will denote it by $\pi_1(X, x)_{\mathcal{M}}$, or briefly, by $\pi_1(X, x)$.

Let $(f, f_0) : (X, x) \to (X', x')$ be a morphism of based objects. The morphism (f, f_0) can be decomposed as

$$
(X, x) \xrightarrow{(f, 1_{D_0})} (X', x'f_0) \xrightarrow{(1_{X'}, f_0)} (X', x').
$$

By applying the functor Π_1, we get a diagram

$$
\pi_1(X, x) \to \pi_1(X', x'f_0) \leftarrow \pi_1(X', x').
$$

Since f_0 is a weak equivalence, the right arrow is an isomorphism and, using its inverse, we get a morphism

$$
\pi_1(f, f_0) : \pi_1(X, x) \to \pi_1(X', x').
$$

One can easily check that this definition makes π_1 a functor from the category of based objects of $\mathcal{M}$ to the category of groups.

If follows from Proposition 6.13 that if f is a weak equivalence, then $\pi_1(f, f_0)$ is an isomorphism. Moreover, by paragraph 6.12, if $D_0 = D'_0$ and $f, g : (X, x) \to (X', x')$ are two morphisms of $D_0 \backslash \mathcal{M}$ inducing the same morphism in $\mathrm{Ho}(D_0 \backslash \mathcal{M})$, then we have

$$
\pi_1(f, 1_{D_0}) = \pi_1(g, 1_{D_0}).
$$

7.4. Loop objects. Let (X, x) be a based object of $\mathcal{M}$ and let P be a path object of X such that $r : X \to P$ is a cofibration. The *loop object* of X based at x (using P) is defined as the pullback of the diagram

$$
D_0 \xrightarrow{(x,x)} X \times X \xleftarrow{(p_1, p_0)} P.
$$

We will denote it by $\Omega_x^P X$. Since every object of $\mathcal{M}$ is fibrant and (p_1, p_0) is a fibration, the cartesian square defining $\Omega_x^P X$ is actually a homotopy cartesian square and $\Omega_x^P X$ is

a model for the homotopy pullback

$$\mathrm{D}_0 \xrightarrow{x} X \xleftarrow{x} \mathrm{D}_0.$$

In particular, the image of $\Omega_x^P X$ in $\mathrm{Ho}(\mathcal{M})$ does not depend on P up to a canonical isomorphism.

One can explicitly construct this canonical isomorphism. Let P' be a second path object of X and let $g : P \to P'$ be a morphism making the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\;r\;} & P & \xrightarrow{(p_1,p_0)} & A \times A \\
{\scriptstyle 1_X}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle 1_{X \times X}} \\
X & \xrightarrow[\;r'\;]{} & P' & \xrightarrow[(p_1',p_0')]{} & A \times A
\end{array}
$$

commute (such a morphism exists by paragraph 6.2). Then the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{D}_0 & \xrightarrow{(x,x)} & X \times X & \xleftarrow{(p_1,p_0)} & P \\
{\scriptstyle 1_{\mathrm{D}_0}}\downarrow & & {\scriptstyle 1_{X \times X}}\downarrow & & \downarrow{\scriptstyle g} \\
\mathrm{D}_0 & \xrightarrow[(x,x)]{} & X \times X & \xleftarrow[(p_1',p_0')]{} & P'
\end{array}
$$

induces a morphism f from $\Omega_x^P X$ to $\Omega_x^{P'} X$. Since the pullbacks defining these objects are homotopy pullbacks and g is a weak equivalence, f is also a weak equivalence. The morphism f hence induces our canonical isomorphism in $\mathrm{Ho}(\mathcal{M})$.

The commutative diagram

$$
\begin{array}{ccccc}
\mathrm{D}_0 & \xrightarrow{\;x\;} & X & \xrightarrow{\;r\;} & P \\
{\scriptstyle 1_{\mathrm{D}_0}}\downarrow & & & & \downarrow{\scriptstyle (p_1,p_0)} \\
\mathrm{D}_0 & & \xrightarrow[(x,x)]{} & & X \times X
\end{array}
$$

induces a morphism $c_x : \mathrm{D}_0 \to \Omega_x^P X$. The loop object $\Omega_x^P X$ is thus endowed with the structure of a based object $(\Omega_x^P X, c_x)$. As above, the image of this object in $\mathrm{Ho}(\mathrm{D}_0 \backslash \mathcal{M})$ does not depend on P up to a canonical isomorphism. For this reason, we will often denote this object by $(\Omega_x X, c_x)$, without reference to P.

Let $(f, f_0) : (X, x) \to (X', x')$ be a morphism of based objects of $\mathcal{M}$. Consider the decomposition of (f, f_0) as

$$(X, x) \xrightarrow{(f, 1_{\mathrm{D}_0})} (X', x' f_0) \xrightarrow{(1_{X'}, f_0)} (X', x').$$

The first morphism can be seen as a morphism of $\mathrm{D}_0 \backslash \mathcal{M}$. Since the objects $\Omega_x X$ and $\Omega_{x' f_0} X'$ are defined as homotopy pullbacks, by the theory of derived functors, the morphism f induces a morphism from $(\Omega_x X, c_x)$ to $(\Omega_{x' f_0} X', c_{x' f_0})$ in $\mathrm{Ho}(\mathrm{D}_0 \backslash \mathcal{M})$.

This morphism can be described explicitly in the following way. Let P (resp. P') be a path object of X (resp. of X') and let $g : P \to P'$ be a morphism making the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\ r\ } & P & \xrightarrow{(p_0,p_1)} & X \times X \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f \times f} \\
Y & \xrightarrow{\ r'\ } & P' & \xrightarrow{(p_1',p_0')} & Y \times Y
\end{array}
$$

commute (such a morphism exists by paragraph 6.2). Then the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{D}_0 & \xrightarrow{(x,x)} & X \times X & \xleftarrow{(p_1,p_0)} & P \\
{\scriptstyle 1_{\mathrm{D}_0}}\downarrow & & \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle g} \\
\mathrm{D}_0 & \xrightarrow{(x',x')f_0} & X' \times X' & \xleftarrow{(p_1',p_0')} & P'
\end{array}
$$

induces a morphism $\Omega_{\mathrm{D}_0} f : \Omega_x X \to \Omega_{x'f_0} X'$ and $(\Omega_{\mathrm{D}_0} f, 1_{\mathrm{D}_0})$ is a morphism of based objects from $(\Omega_x X, c_x)$ to $(\Omega_{x'f_0} X', c_{x'f_0})$. This morphism induces our canonical morphism in $\mathrm{Ho}(\mathrm{D}_0 \backslash \mathcal{M})$.

The second morphism $(1_{X'}, f_0)$ gives rise to a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{D}_0 & \xrightarrow{(x',x')f_0} & X' \times X' & \xleftarrow{(p_1',p_0')} & P' \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle 1_{X' \times X'}} & & \downarrow{\scriptstyle 1_{P'}} \\
\mathrm{D}_0' & \xrightarrow{(x',x')} & X' \times X' & \xleftarrow{(p_1',p_0')} & P' \quad .
\end{array}
$$

This diagram induces a canonical morphism $\Omega_{f_0} X' : \Omega_{x'f_0} X' \to \Omega_{x'} X'$, and $(\Omega_{f_0} X', f_0)$ is a morphism of based objects from $(\Omega_{x'f_0} X', c_{x'f_0})$ to $(\Omega_{x'} X', c_{x'})$.

We set

$$
\Omega_{f_0} f = \Omega_x X \xrightarrow{\Omega_{\mathrm{D}_0} f} \Omega_{x'f_0} X' \xrightarrow{\Omega_{f_0} X'} \Omega_{x'} X'.
$$

This morphism induces a canonical morphism in $\mathrm{Ho}(\mathcal{M})$. Moreover, $(\Omega_{f_0} f, f_0)$ is a morphism of based objects from $(\Omega_x X, c_x)$ to $(\Omega_{x'} X', c_{x'})$

7.5. Higher homotopy groups. Let (X, x) be a based object of $\mathcal{M}$. Since $(\Omega_x X, c_x)$ is well-defined, up to a canonical isomorphism coming from $\mathrm{D}_0 \backslash \mathcal{M}$, as an object $\mathrm{Ho}(\mathrm{D}_0 \backslash \mathcal{M})$, the group $\pi_1(\Omega_x X, c_x)$ is well-defined by the last point of paragraph 7.3.

Let $(f, f_0) : (X, x) \to (X', x')$ be a morphism of based objects of $\mathcal{M}$. We claim that the morphism $\pi_1(\Omega_x f, f_0)$ is well-defined. Recall that by definition, $\Omega_{f_0} f$ is the composition of

$$
\Omega_x X \xrightarrow{\Omega_{\mathrm{D}_0} f} \Omega_{x'f_0} X' \xrightarrow{\Omega_{f_0} X'} \Omega_{x'} X',
$$

where the first morphism is well-defined as a morphism of $\mathrm{Ho}(\mathrm{D}_0 \backslash \mathcal{M})$ and the second morphism is well-defined as a morphism of $\mathcal{M}$. It follows that $\pi_1(\Omega_{f_0} f, f_0)$ is the composition of

$$
\pi_1(\Omega_x X, c_x) \xrightarrow{\pi_1(\Omega_{\mathrm{D}_0} f, 1_{\mathrm{D}_0})} \pi_1(\Omega_{x'f_0} X', x'f_0) \xrightarrow{\pi_1(\Omega_{f_0} X', f_0)} \pi_1(\Omega_{x'} X', c_{x'}),
$$

where both morphisms are well-defined (the left one is well-defined by the last point of paragraph 7.3). One easily checks that $\pi_1(\Omega_{f_0} f, f_0)$ is functorial in (f, f_0).

For $n \geq 2$, we define, by induction on n, the *n-th homotopy group* of (X, x) as

$$\pi_n(X, x) = \pi_{n-1}(\Omega_x X, c_x).$$

It follows from the above discussion that π_n is a well-defined functor from the category of based objects of $\mathcal{M}$ to the category of (abelian) groups.

Let $(f, f_0) : (X, x) \to (X', x')$ be a morphism of based objects of $\mathcal{M}$. It is immediate, by induction on $n \geq 1$, that if f is a weak equivalence, then

$$\pi_n(f, f_0) : \pi_n(X, x) \to \pi_n(X', x')$$

is an isomorphism.

7.6. Functoriality of $[\mathrm{D}_0, \Omega_x X]$**.** Let $(f, f_0) : (X, x) \to (X', x')$ be a morphism of based objects of $\mathcal{M}$. The morphisms $\Omega_{f_0} f : \Omega_x X \to \Omega_{x'} X'$ and $f_0 : \mathrm{D}_0 \to \mathrm{D}_0'$ induce a diagram

$$[\mathrm{D}_0, \Omega_x X] \to [\mathrm{D}_0, \Omega_{x'} X'] \leftarrow [\mathrm{D}_0', \Omega_{x'} X'].$$

Since f_0 is a weak equivalence, the right arrow is a bijection. Using the inverse of this bijection, we get a map

$$[f, f_0] : [\mathrm{D}_0, \Omega_x X] \to [\mathrm{D}_0', \Omega_{x'} X'].$$

Note that the inverse of the bijection is induced by any inverse of f_0 up to homotopy (remember that D_0 and D_0' are fibrant and cofibrant). The morphism $[f, f_0]$ is easily seen to be functorial in (f, f_0).

Proposition 7.7. *Let* $(X, x : \mathrm{D}_0 \to X)$ *be a based object of* $\mathcal{M}$*. Then there exists a canonical bijection*

$$\pi_1(X, x) \cong [\mathrm{D}_0, \Omega_x X] = \pi_0(\Omega_x X),$$

natural in (X, x)*.*

Proof. Let P be a path object of X such that $r : P \to X$ is a cofibration. Recall that

$$\Omega_x X = \Omega_x^P X = (P, (p_1, p_0)) \times_{X \times X} ((x, x), \mathrm{D}_0).$$

If Y is an object of $\mathcal{M}$, a morphism $f : Y \to \Omega_x X$ of $\mathcal{M}$ is hence given by a pair of morphisms $u : Y \to P$ and $v : Y \to \mathrm{D}_0$ of $\mathcal{M}$ such that $(p_1, p_0)u = (x, x)v$. We will denote by (u, v) this morphism.

By the dual of Proposition 6.9, we have a canonical bijection

$$\pi_1(X, x) \cong [(\mathrm{D}_0, (x, x)), (P, (p_1, p_0))]_{\mathcal{M}/(X \times X)}.$$

Let

$$m : [(\mathrm{D}_0, (x, x)), (P, (p_1, p_0))]_{\mathcal{M}/(X \times X)} \to [\mathrm{D}_0, \Omega_x X]$$

be the map sending a morphism

$$u : (\mathrm{D}_0, (x, x)) \to (P, (p_1, p_0))$$

to the morphism

$$(u, 1_{\mathrm{D}_0}) : \mathrm{D}_0 \to \Omega_x X.$$

Let us show that this map is well-defined. Let C be a cylinder object of D_0. Then $(C, (x, x)s)$ is a cylinder object of $(\mathrm{D}_0, (x, x))$ in $\mathcal{M}/(X \times X)$. Thus, a left homotopy

$h : (C, (x, x)s) \to (P, (p_1, p_0))$ of $\mathcal{M}/(X \times X)$ from a morphism u to a morphism u' induces a left homotopy $(h, s) : C \to \Omega_x X$ of $\mathcal{M}$ from the morphism $(u, 1_{\mathrm{D}_0})$ to the morphism $(u', 1_{\mathrm{D}_0})$.

Let us show that the map m is surjective. Let $(u, v) : \mathrm{D}_0 \to \Omega_x X$ be a morphism of $\mathcal{M}$. We have to show that (u, v) is left homotopic to $(u', 1_{\mathrm{D}_0})$ in $\mathcal{M}$ for some morphism $u' : (\mathrm{D}_0, (x, x)) \to (P, (p_1, p_0))$ of $\mathcal{M}/(X \times X)$. Consider the commutative square

$$
\begin{array}{ccc}
\mathrm{D}_0 \amalg \mathrm{D}_0 & \xrightarrow{(1_{\mathrm{D}_0}, v)} & \mathrm{D}_0 \\
{\scriptstyle (i_1, i_0)} \downarrow & & \downarrow \\
C & \longrightarrow & *
\end{array} \quad .
$$

Since D_0 is fibrant and weakly contractible, this square admits a lifting $h : C \to \mathrm{D}_0$. Consider now the commutative square

$$
\begin{array}{ccc}
\mathrm{D}_0 & \xrightarrow{\;\;u\;\;} & P \\
{\scriptstyle i_0} \downarrow & & \downarrow {\scriptstyle (p_1, p_0)} \\
C & \xrightarrow[h]{} \mathrm{D}_0 \xrightarrow[(x,x)]{} & X \times X
\end{array} \quad .
$$

Since D_0 is cofibrant, i_0 is a trivial cofibration and the square admits a lifting $k : C \to P$. Set $u' = k i_1$. The morphism u' induces a morphism $u' : (\mathrm{D}_0, (x, x)) \to (P, (p_1, p_0))$ of $\mathcal{M}/(X \times X)$, and the morphisms h and k induce a morphism $(k, h) : C \to \Omega_x X$ which is a left homotopy of $\mathcal{M}$ from (u, v) to $(u', 1_{\mathrm{D}_0})$.

Let us now show that m is injective. Let $u, u' : (\mathrm{D}_0, (x, x)) \to (P, (p_1, p_0))$ be two morphisms of $\mathcal{M}/(X \times X)$. Suppose $(k, h) : C \to \Omega_x X$ is a left homotopy of $\mathcal{M}$ from $(u, 1_{\mathrm{D}_0})$ to $(u', 1_{\mathrm{D}_0})$. We have to show that u and u' are left homotopic in $\mathcal{M}/(X \times X)$. Let C' be a cylinder object of C in $\mathcal{M}$. Consider the commutative square

$$
\begin{array}{ccc}
C \amalg C & \xrightarrow{(s, h)} & \mathrm{D}_0 \\
{\scriptstyle (i'_1, i'_0)} \downarrow & & \downarrow \\
C' & \longrightarrow & *
\end{array} \quad .
$$

This square admits a lifting $H : C' \to \mathrm{D}_0$ for the same reasons as above. Consider now the commutative square

$$
\begin{array}{ccc}
C & \xrightarrow{\;\;k\;\;} & P \\
{\scriptstyle i'_0} \downarrow & & \downarrow {\scriptstyle (p_1, p_0)} \\
C' & \xrightarrow[H]{} \mathrm{D}_0 \xrightarrow[(x,x)]{} & X \times X
\end{array} \quad .
$$

Since D_0 is cofibrant, C is cofibrant and $i'_0 : C \to C'$ is a trivial cofibration. The square hence admits a lifting $K : C' \to P$. The morphism $k' = K i'_1 : C \to P$ induces a morphism $k' : (C, (x, x)s) \to (P, (p_1, p_0))$ which is a left homotopy of $\mathcal{M}/(X \times X)$ from $k' i_0$ to $k' i_1$. But we have

$$
k' i_0 = K i'_1 i_0 = K i'_0 i_0 = k i_0 = u,
$$

and a similar calculation shows that $k'i_1 = u'$.

Finally, let us show the naturality of this bijection. Let $(f, f_0) : (X, x) \to (X, x')$ be a morphism of based objects. By the decomposition of such a morphism given in paragraph 7.3, it suffices to show the result when f or f_0 is an identity. Suppose first that f_0 is the identity. Let P (resp. P') be a path object of X (resp. of X') and assume that $r : X \to P$ is a cofibration. Let $g : P \to P'$ be a morphism making the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\ r\ } & P & \xrightarrow{(p_1, p_0)} & X \times X \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f \times f} \\
X' & \xrightarrow[\ r'\ ]{} & P' & \xrightarrow[(p_1', p_0')]{} & X' \times X'
\end{array}
$$

commute. Consider the naturality square

$$
\begin{array}{ccc}
[(D_0, (x, x)), (P, (p_1, p_0))]_{\mathcal{M}/(X \times X)} & \longrightarrow & [D_0, \Omega_x X] \\
\downarrow & & \downarrow \\
[(D_0, (x, x)), (P', (p_1', p_0'))]_{\mathcal{M}/(X \times X)} & \longrightarrow & [D_0, \Omega_{x'} X'] \quad .
\end{array}
$$

The right vertical map is induced by g by definition and the left vertical map is induced by g by Lemma 6.11. The square is hence commutative.

Suppose now f is the identity. The vertical maps in the naturality square

$$
\begin{array}{ccc}
[(D_0, (x, x)), (P, (p_1, p_0))]_{\mathcal{M}/(X \times X)} & \longrightarrow & [D_0, \Omega_x X] \\
\downarrow & & \downarrow \\
[(D_0', (x, x)), (P, (p_1, p_0))]_{\mathcal{M}/(X \times X)} & \longrightarrow & [D_0', \Omega_x X]
\end{array}
$$

are both induced by an inverse of f_0 up to homotopy and the square is hence commutative. $\qquad\square$

7.8. A description of the composition of homotopies. Let X be an object of $\mathcal{M}$. Choose D_0 a cofibrant replacement of the terminal object of $\mathcal{M}$ and D_1 a cylinder object of D_0. Denote by

$$
D_0 \amalg D_0 \xrightarrow{(\tau_1, \sigma_1)} D_1 \xrightarrow{\ \kappa_0\ } D_0
$$

the associated factorization.

Let $x, x', x'' : D_0 \to X$ be three morphisms of $\mathcal{M}$. Every element of $\pi_1(D_0, X; x, x')$ can be represented by a homotopy $l : D_1 \to X$ from x to x'. Similarly, an element of $\pi_1(D_0, X; x', x'')$ can be represented by a homotopy $l' : D_1 \to X$ from x' to x''. Let l and l' be such homotopies. We will describe a homotopy $D_1 \to X$ from x to x'' representing the composition of l and l'.

Let

$$
D_1 \amalg_{D_0} D_1 = (D_1, \sigma_1) \amalg_{D_0} (\tau_1, D_1)
$$

be the cylinder considered in paragraph 6.6. Denote by

$$
\varepsilon_1, \varepsilon_2 : D_1 \to D_1 \amalg_{D_0} D_1
$$

the two canonical morphisms and consider the commutative square

$$
\begin{array}{ccc}
D_0 \amalg D_0 & \xrightarrow{(\varepsilon_1\tau_1,\varepsilon_2\sigma_1)} & D_1 \amalg_{D_0} D_1 \\
{\scriptstyle(\tau_1,\sigma_1)}\Big\downarrow & & \Big\downarrow \\
D_1 & \xrightarrow{\hspace{3cm}} & *
\end{array}
$$

The left vertical morphism is a cofibration by definition. Since $D_1 \amalg_{D_0} D_1$ is a cylinder object of a weakly contractible object, it is also weakly contractible. Moreover, by assumption on the model category $\mathcal{M}$, every object is fibrant. The square hence admits a lifting, i.e., there exists a morphism

$$\nabla_1 : D_1 \to D_1 \amalg_{D_0} D_1$$

such that

$$\nabla_1\sigma_1 = \varepsilon_2\sigma_1 \quad \text{and} \quad \nabla_1\tau_1 = \varepsilon_1\tau_1.$$

The morphism $(l',l)\nabla_1$ is the announced homotopy from x to x''.

Proposition 7.9. *With the notation of the above paragraph, the homotopy $(l,l')\nabla_1$ corresponds under the bijection of Proposition 6.9, to the composition of l and l'. In other words, the homotopies*

$$(l',l) : D_1 \amalg_{D_0} D_1 \to X \quad \text{and} \quad (l',l)\nabla_1 : D_1 \to X$$

are left 2-homotopic.

Proof. Let D be a 2-cylinder object of the cylinders $D_1 \amalg_{D_0} D_1$ and D_1. Consider the commutative square

$$
\begin{array}{ccc}
D_1 \amalg_{D_0} (D_1 \amalg_{D_0} D_1) & \xrightarrow{(\nabla_1, 1_{D_1 \amalg_{D_0} D_1})} & D_1 \amalg_{D_0} D_1 \\
{\scriptstyle(j_1,j_0)}\Big\downarrow & & \Big\downarrow \\
D & \xrightarrow{\hspace{3cm}} & *
\end{array}
$$

The left vertical morphism is a cofibration by definition and the object $D_1 \amalg_{D_0} D_1$ is fibrant and weakly contractible. The square hence admits a lifting $H : D \to D_1 \amalg_{D_0} D_1$ and $(l',l)H$ is the desired left 2-homotopy from (l',l) to $(l',l)\nabla_1$, $\qquad\square$

8. Comparison of Quillen and Grothendieck homotopy groups

8.1. In this section, we fix a formal coherator C, a model category $\mathcal{M}$ in which every object is fibrant, and a fibrant and weakly contractible functor $F : \mathbb{G} \to \mathcal{M}$. By paragraph 5.14, such a functor induces a functor

$$\Pi_{C,K} : \mathcal{M} \to \infty\text{-}\mathcal{G}\mathrm{pd}_C,$$

which depends on a globular functor $K : C \to \mathcal{M}$. From now on, we will denote this functor by Π_∞ and, if X is an object of $\mathcal{M}$, we will call $\Pi_\infty(X)$ the fundamental ∞-groupoid of X. Recall that the set of n-arrows of this fundamental ∞-groupoid is the set of morphisms from D_n to X in $\mathcal{M}$.

We recall some notation from paragraph 5.13. For $n \geq 0$, we have an object S^{n-1} of $\mathcal{M}$ and a morphism $i_n : \mathrm{S}^{n-1} \to \mathrm{D}_n$. For $n = 0$, the morphism i_0 is the unique morphism $\varnothing \to \mathrm{D}_0$ and, for $n \geq 1$, we have

$$i_n = (\tau_n, \sigma_n) : \mathrm{S}^{n-1} = \mathrm{D}_{n-1} \amalg_{\mathrm{S}^{n-2}} \mathrm{D}_{n-1} \to \mathrm{D}_n.$$

The hypothesis on F exactly means that i_n is a cofibration and that the D_n is weakly contractible.

For $n \geq 0$, we will denote by $j_{n+1} : \mathrm{S}^{n-1} \to \mathrm{D}_{n+1}$ the composition of the canonical morphism $\mathrm{S}^{n-1} \to \mathrm{S}^n = \mathrm{D}_n \amalg_{\mathrm{S}^{n-1}} \mathrm{D}_n$ followed by i_{n+1}. Note that we have

$$j_{n+1} = (\tau_{n+1}\tau_n, \sigma_{n+1}\sigma_n) : \mathrm{S}^{n-1} = \mathrm{D}_{n-1} \amalg_{\mathrm{S}^{n-2}} \mathrm{D}_{n-1} \to \mathrm{D}_{n+1}.$$

We will also fix a choice of morphisms

$$\kappa_n : \mathrm{D}_{n+1} \to \mathrm{D}_n, \quad n \geq 0,$$

of C such that

$$\kappa_n \sigma_{n+1} = 1_{\mathrm{D}_n} \quad \text{and} \quad \kappa_n \tau_{n+1} = 1_{\mathrm{D}_n}.$$

These morphisms are weak equivalences since the D_n's are weakly contractible. Moreover, they induce a choice of units $k_n : G_n \to G_{n-1}$ for every ∞-groupoid G of type C.

Lemma 8.2.

(1) *The object D_0 is a cofibrant replacement of the terminal object of $\mathcal{M}$.*
(2) *For every $n \geq 0$, the object $(\mathrm{D}_{n+1}, j_{n+1})$ is a cylinder object of (D_n, i_n) in the category $\mathrm{S}^{n-1} \backslash \mathcal{M}$ for the factorization*

$$(\mathrm{S}^n, l_n) \xrightarrow{\ i_{n+1}\ } (\mathrm{D}_{n+1}, j_{n+1}) \xrightarrow{\ \kappa_n\ } (\mathrm{D}_n, i_n) \quad,$$

where $l_n : \mathrm{S}^{n-1} \to \mathrm{S}^n = \mathrm{D}_n \amalg_{\mathrm{S}^{n-1}} \mathrm{D}_n$ is the canonical morphism.

Proof. The first assertion is true by definition. Let us prove the second one. The equalities

$$i_{n+1}l_n = j_{n+1}, \quad \kappa_n j_{n+1} = \kappa_n(\tau_{n+1}\tau_n, \sigma_{n+1}\sigma_n) = (\tau_n, \sigma_n) = i_n,$$

and

$$\kappa_n i_{n+1} = \kappa_n(\tau_{n+1}, \sigma_{n+1}) = (1_{\mathrm{D}_n}, 1_{\mathrm{D}_n}),$$

show that we indeed have a factorization of the codiagonal of (D_n, i_n) in $\mathrm{S}^{n-1} \backslash \mathcal{M}$. Moreover, i_{n+1} is a cofibration and we have already noticed that κ_n is a weak equivalence. $\square$

Proposition 8.3. *Let X be an object of $\mathcal{M}$ and let G be its fundamental ∞-groupoid.*
(1) *Two objects $x, y : \mathrm{D}_0 \to X$ of G are homotopic as objects of G if and only if $x, y : \mathrm{D}_0 \to X$ are equal as morphisms of $\mathrm{Ho}(\mathcal{M})$. In particular, we have a canonical bijection*

$$\pi_0(\Pi_\infty(X)) \to \pi_0(X),$$

natural in X.
(2) *Let $n \geq 1$ and let $x, y : \mathrm{D}_{n-1} \to X$ be two parallel $(n-1)$-arrows of G. Two n-arrows $u, v : \mathrm{D}_n \to X$ from x to y are homotopic as n-arrows of G if and only if*

$u, v : (\mathrm{D}_n, i_n) \to (X, (y, x))$ *are equal as morphisms of* $\mathrm{Ho}(\mathrm{S}^{n-1} \backslash \mathcal{M})$. *In particular, we have a canonical bijection*

$$\pi_n(\Pi_\infty(X), x, y) \to [(\mathrm{D}_n, i_n), (X, (y, x))]_{\mathrm{S}^{n-1} \backslash \mathcal{M}},$$

natural in X.

Proof.

(1) By definition, two objects $x, y : \mathrm{D}_0 \to X$ of G are homotopic if there exists a 1-arrow $h : x \to y$ in G, i.e., a morphism $h : \mathrm{D}_1 \to X$ of $\mathcal{M}$ such that $h\sigma_1 = x$ and $h\tau_1 = y$. By the previous lemma, D_1 is a cylinder object of D_0 and so h is a left homotopy from x to y in $\mathcal{M}$. The result then follows from the fact that D_0 is cofibrant and X is fibrant.

(2) By definition, the n-arrows $u, v : \mathrm{D}_n \to X$ are homotopic if there exists an $(n+1)$-arrow $h : u \to v$ in G, i.e., a morphism $h : \mathrm{D}_{n+1} \to X$ of $\mathcal{M}$ such that $h\sigma_{n+1} = u$ and $h\tau_{n+1} = v$. Such an h induces a morphism $h : (\mathrm{D}_{n+1}, j_{n+1}) \to (X, (y, x))$. But by the previous lemma, $(\mathrm{D}_{n+1}, j_{n+1})$ is a cylinder object of (D_n, i_n) in $\mathrm{S}^{n-1} \backslash \mathcal{M}$ and so h is a left homotopy between $u, v : (\mathrm{D}_n, i_n) \to (X, (y, x))$ in $\mathrm{S}^{n-1} \backslash \mathcal{M}$. The result then follows from the fact that i_n is a cofibration and X is fibrant. $\square$

Proposition 8.4. *Let* X *be an object of* $\mathcal{M}$. *There is a canonical isomorphism of groupoids*

$$\varpi_1(\Pi_\infty(X)) \cong \Pi_1(\mathrm{D}_0, X),$$

natural in X.

Proof. Let $x, y : \mathrm{D}_0 \to X$ be two objects of $\varpi_1(\Pi_\infty(X))$. By the previous proposition, we have

$$\mathsf{Hom}_{\varpi_1(\Pi_\infty(X))}(x, y) \cong [(\mathrm{D}_1, i_1), (X, (y, x))]_{\mathrm{S}^0 \backslash \mathcal{M}}.$$

Hence by Proposition 6.9, the underlying graphs of $\varpi_1(\Pi_\infty(X))$ and of $\Pi_1(\mathrm{D}_0, X)$ are canonically isomorphic. This isomorphism is obviously natural in X. Moreover, by definition, the composition of $\varpi_1(\Pi_\infty(X))$ is induced by any morphism

$$\nabla_1 : \mathrm{D}_1 \to \mathrm{D}_1 \amalg_{\mathrm{D}_0} \mathrm{D}_1$$

such that

$$\nabla_1 \sigma_1 = \varepsilon_2 \sigma_1 \quad \text{and} \quad \nabla_1 \tau_1 = \varepsilon_1 \tau_1.$$

The result thus follows from Proposition 7.9. $\square$

Proposition 8.5. *Let* X *be an object of* $\mathcal{M}$. *Let* $n \geq 2$ *and let* $u : x \to y$ *be an* $(n-1)$*-arrow of the fundamental* ∞*-groupoid of* X. *We have a canonical isomorphism of groups*

$$\pi_n(\Pi_\infty(X), u) \cong \pi_1((X, (y, x)), u)_{\mathrm{S}^{n-2} \backslash \mathcal{M}},$$

natural in X.

Proof. We have the following series of natural bijections of sets:

$$
\begin{aligned}
\pi_n(\Pi_\infty(X), u) &\cong [(\mathrm{D}_n, i_n), (X, (u, u))]_{\mathrm{S}^{n-1} \backslash \mathcal{M}} && \text{(by Proposition 8.3)} \\
&\cong \pi_1((\mathrm{D}_{n-1}, i_{n-1}), (X, (y, x)); u, u)_{\mathrm{S}^{n-2} \backslash \mathcal{M}} && \text{(see below)} \\
&= \pi_1((X, (y, x)), u)_{\mathrm{S}^{n-2} \backslash \mathcal{M}}.
\end{aligned}
$$

Let us justify the second bijection. By Lemma 8.2, the object (D_n, j_n) is a cylinder object of (D_{n-1}, i_{n-1}) in $S^{n-2}\backslash\mathcal{M}$. This bijection thus comes from Proposition 6.9 applied to the model category $S^{n-2}\backslash\mathcal{M}$.

Furthermore, the composition of $\pi_n(\Pi_\infty(X), u)$ is induced by any morphism

$$\nabla_n : D_n \to D_n \amalg_{D_{n-1}} D_n$$

such that

$$\nabla_n \sigma_n = \varepsilon_2 \sigma_n \quad \text{and} \quad \nabla_n \tau_n = \varepsilon_1 \tau_n.$$

Denote by $k_n : S^{n-2} \to D_n \amalg_{D_{n-1}} D_n$ the composition of $i_{n-1} : S^{n-2} \to D_{n-1}$ followed by the canonical morphism $D_{n-1} \to D_n \amalg_{D_{n-1}} D_n$. We have

$$(D_n \amalg_{D_{n-1}} D_n, k_n) = (D_n, j_n) \amalg_{(D_{n-1}, i_{n-1})} (D_n, j_n),$$

where the amalgamated sum is taken in $S^{n-2}\backslash\mathcal{M}$. Moreover, a morphism ∇_n as above induces a morphism $\nabla_n : (D_n, j_n) \to (D_n \amalg_{D_{n-1}} D_n, k_n)$. It thus follows from Proposition 7.9 that the bijection we have defined is a morphism of groups. $\square$

Proposition 8.6. *Let X be an object of $\mathcal{M}$ and let x be an object of the fundamental ∞-groupoid of X. For every $n \geq 2$, there is a canonical isomorphism of groups*

$$\pi_n(\Pi_\infty(X), x) \cong \pi_{n-1}(\Pi_\infty(\Omega_x X), c_x),$$

natural in X.

Proof. We have the following series of natural isomorphisms:

$$\begin{aligned}
\pi_n(\Pi_\infty(X), x) &= \pi_n(\Pi_\infty(X), k_{n-1}^0(x)) \\
&\cong \pi_1((X, (t_{n-1}k_{n-1}^0(x), s_{n-1}k_{n-1}^0(x))), k_{n-1}^0(x))_{S^{n-2}\backslash\mathcal{M}} \\
&\quad \text{(by the previous proposition)} \\
&= \pi_1((X, (k_{n-2}^0(x), k_{n-2}^0(x))), k_{n-1}^0(x))_{S^{n-2}\backslash\mathcal{M}} \\
&\cong [(D_{n-1}, i_{n-1}), \Omega_{k_{n-1}^0(x)}(X, (k_{n-2}^0(x), k_{n-2}^0(x)))]_{S^{n-2}\backslash\mathcal{M}} \\
&\quad \text{(by Proposition 7.7)} \\
&\cong [(D_{n-1}, i_{n-1}), (\Omega_{k_{n-1}^0(x)}X, (c_{k_{n-2}^0(x)}, c_{k_{n-2}^0(x)}))]_{S^{n-2}\backslash\mathcal{M}} \\
&\cong [(D_{n-1}, i_{n-1}), (\Omega_x X, (c_{k_{n-2}^0(x)}, c_{k_{n-2}^0(x)}))]_{S^{n-2}\backslash\mathcal{M}} \\
&\quad \text{(see below)} \\
&\cong \pi_{n-1}(\Pi_\infty(\Omega_x X), c_{k_{n-2}^0(x)}) \\
&\quad \text{(by Proposition 8.3)} \\
&= \pi_{n-1}(\Pi_\infty(\Omega_x X), k_{n-2}^0(c_x)) \\
&= \pi_{n-1}(\Pi_\infty(\Omega_x X), c_x).
\end{aligned}$$

To end the proof, it suffices to show that we have a canonical weak equivalence

$$(\Omega_{k_{n-1}^0(x)}X, (c_{k_{n-2}^0(x)}, c_{k_{n-2}^0(x)})) \to (\Omega_x X, (c_{k_{n-2}^0(x)}, c_{k_{n-2}^0(x)}))$$

in $S^{n-2}\backslash\mathcal{M}$. Consider the commutative diagram

$$
\begin{array}{ccccc}
D_{n-1} & \xrightarrow{(k_{n-1}^0(x),k_{n-1}^0(x))} & X \times X & \xleftarrow{(p_1,p_0)} & P \\
{\scriptstyle \kappa_{n-1}^0}\downarrow & & {\scriptstyle 1_{X\times X}}\downarrow & & \downarrow{\scriptstyle 1_P} \\
D_0 & \xrightarrow{(x,x)} & X \times X & \xleftarrow{(p_1,p_0)} & P
\end{array} \; ,
$$

where P is a path object of X in $\mathcal{M}$. This diagram induces a morphism

$$\Omega_{k_{n-1}^0(x)}X \to \Omega_x X$$

of $\mathcal{M}$. Since κ_{n-1}^0 is a weak equivalence and the pullbacks defining loop objects are homotopy pullbacks, this morphism is a weak equivalence. Moreover, it induces a morphism

$$(\Omega_{k_{n-1}^0(x)}X, (c_{k_{n-2}^0(x)}, c_{k_{n-2}^0(x)})) \to (\Omega_x X, (c_{k_{n-2}^0(x)}, c_{k_{n-2}^0(x)}))$$

in $S^{n-2}\backslash\mathcal{M}$. This is our desired weak equivalence. $\qquad\square$

Theorem 8.7. *Let X be an object of $\mathcal{M}$.*

- *There is a canonical bijection*

$$\pi_0(\Pi_\infty(X)) \cong \pi_0(X),$$

  natural in X.
- *Let $n \geq 1$ and let x be an object of the fundamental ∞-groupoid of X. There is a canonical isomorphism of groups*

$$\pi_n(\Pi_\infty(X), x) \cong \pi_n(X, x),$$

  natural in X.

Proof. Let us prove the result by induction on $n \geq 0$. For $n = 0$ and $n = 1$, the result is a direct consequence of Proposition 8.4. For $n \geq 2$, we have

$$
\begin{aligned}
\pi_n(\Pi_\infty(X), x) &\cong \pi_{n-1}(\Pi_\infty(\Omega_x X), c_x) && \text{(by the previous proposition)} \\
&\cong \pi_{n-1}(\Omega_x X, c_x) && \text{(by induction hypothesis)} \\
&= \pi_n(X, x),
\end{aligned}
$$

thereby proving the theorem. $\qquad\square$

Corollary 8.8. *If X is an object of $\mathcal{M}$, the homotopy groups of $\Pi_\infty(X)$ depend only on X. In particular, they do not depend on the choice of the formal coherator C.*

Corollary 8.9. *Let $f : X \to Y$ be a morphism of $\mathcal{M}$. Then $\Pi_\infty(f)$ is a weak equivalence of ∞-groupoids of type C if and only if the following conditions are satisfied:*

- *the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection;*
- *for every $n \geq 1$ and every base point $x : D_0 \to X$, the morphism $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism.*

Proof. This follows immediately from the naturality of the isomorphisms of the above theorem. $\qquad\square$

Corollary 8.10. *Let f be a weak equivalence of $\mathcal{M}$. Then $\Pi_\infty(f)$ is a weak equivalence of ∞-groupoids of type C.*

Proof. This follows immediately from the previous corollary and the fact that weak equivalences of $\mathcal{M}$ induce isomorphisms on homotopy groups. $\qquad\qquad\square$

Corollary 8.11. *Let $\mathcal{M}$ be $\mathcal{T}op$ endowed with its usual model category structure. A map f is a weak equivalence of topological spaces if and only if $\Pi_\infty(f)$ is a weak equivalence of ∞-groupoids of type C.*

Proof. This follows immediately from Corollary 8.9. $\qquad\qquad\square$

Remark 8.12. This corollary is proved directly in Section 4.4 of [1] (see in particular Corollary 4.4.11).

8.13. The functor $\overline{\Pi_\infty}$. Let $\mathcal{W}_{\mathcal{T}op}$ be the class of weak equivalences of topological spaces and let $\mathcal{W}_{\infty\text{-}\mathcal{G}\mathrm{pd}_C}$ be the class of weak equivalences of ∞-groupoids of type C. By the previous corollary, the functor Π_∞ sends $\mathcal{W}_{\mathcal{T}op}$ into $\mathcal{W}_{\infty\text{-}\mathcal{G}\mathrm{pd}_C}$. This functor thus induces a functor

$$\overline{\Pi_\infty} : \mathsf{Hot} = \mathcal{T}op[\mathcal{W}_{\mathcal{T}op}^{-1}] \to \mathrm{Ho}(\infty\text{-}\mathcal{G}\mathrm{pd}_C) = \infty\text{-}\mathcal{G}\mathrm{pd}_C[\mathcal{W}_{\infty\text{-}\mathcal{G}\mathrm{pd}_C}^{-1}].$$

We can now state a precise version of Grothendieck's conjecture:

Conjecture 8.14 (Grothendieck)**.** *If C is a coherator, the functor $\overline{\Pi_\infty}$ is an equivalence of categories.*

References

1. D. Ara, *Sur les ∞-groupoïdes de Grothendieck et une variante ∞-catégorique*, Ph.D. thesis, Université Paris 7, 2010.
2. ———, *The groupoidal analogue $\widetilde{\Theta}$ to Joyal's category Θ is a test category*, Appl. Categ. Structures **20** (2012), no. 6, 603–649.
3. ———, *Strict ∞-groupoids are Grothendieck ∞-groupoids*, preprint, 2012.
4. D. Ara and F. Métayer, *The Brown-Golasiński model structure on strict ∞-groupoids revisited*, Homology Homotopy Appl. **13** (2011), no. 1, 121–142.
5. A. Bastiani and C. Ehresmann, *Categories of sketched structures*, Cahiers Topologie Géom. Différentielle **13** (1972), 105–214.
6. M. A. Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Adv. Math. **136** (1998), no. 1, 39–103.
7. C. Berger, *A cellular nerve for higher categories*, Adv. Math. **169** (2002), no. 1, 118–175.
8. ———, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), no. 1, 230–270.
9. D. Dugger, *Combinatorial model categories have presentations*, Adv. Math. **164** (2001), no. 1, 177–201.
10. R. Gordon, A. J. Power, and Ross Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. **117** (1995), no. 558, vi+81.
11. A. Grothendieck, *Pursuing stacks*, Manuscript, 1983, to be published in Documents Mathématiques.
12. P. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, 2003.
13. A. Joyal, *Disks, duality and Θ-categories*, preprint, 1997.
14. M. Makkai and M. Zawadowski, *Duality for simple ω-categories and disks*, Theory Appl. Categ. **8** (2001), 114–243.
15. G. Maltsiniotis, *Infini groupoïdes non stricts, d'après Grothendieck*, preprint, 2007.

16. _____, *Infini catégories non strictes, une nouvelle définition*, preprint, 2007.
17. _____, *Grothendieck ∞-groupoids and still another definition of ∞-categories*, preprint, 2010.
18. T. Nikolaus, *Algebraic models for higher categories*, Indag. Math. (N.S.) **21** (2011), no. 1–2, 52–75.
19. D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, vol. 43, Springer-Verlag, 1967.
20. C. Simpson, *Homotopy types of strict 3-groupoids*, arXiv:math/9810059v1 [math.CT], 1998.
21. _____, *Homotopy theory of higher categories*, New mathematical monographs, vol. 19, Cambridge University Press, 2011.

DIMITRI ARA, INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UNIVERSITÉ PARIS DIDEROT – PARIS 7, CASE 7012, BÂTIMENT CHEVALERET, 75205 PARIS CEDEX 13, FRANCE
  Email address: `ara@math.jussieu.fr`
  URL: `http://people.math.jussieu.fr/~ara/`