THE FOLK MODEL CATEGORY STRUCTURE ON STRICT ω-CATEGORIES IS MONOIDAL

by

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Abstract. — We prove that the folk model category structure on the category of strict ω-categories, introduced by Lafont, Métayer and Worytkiewicz, is monoidal, first, for the Gray tensor product and, second, for the join of ω-categories, introduced by the first author and Maltsiniotis. We moreover show that the Gray tensor product induces, by adjunction, a tensor product of strict (m, n)-categories and that this tensor product is also compatible with the folk model category structure. In particular, we get a monoidal model category structure on the category of strict ω-groupoids. We prove that this monoidal model category structure satisfies the monoid axiom, so that the category of Gray monoids, studied by the second author, bears a natural model category structure.

Introduction

The category $\omega\text{-}\mathcal{C}at$ of strict ω-categories, that we shall simply call ω-categories in this paper, is endowed with a model category structure, introduced by Lafont, Métayer and Worytkiewicz [**15**], known as the *folk* model category structure. The weak equivalences of this structure are the equivalences of ω-categories, higher dimensional generalization of the equivalences of categories or of 2-categories; the cofibrant objects are the ω-categories that are free in the sense of polygraphs [**17**]. This model category structure, which is also called the canonical model category structure, is in some sense intrinsic to the notion of ω-categories.

On the other hand, the category $\omega\text{-}\mathcal{C}at$ is endowed with two non-trivial monoidal category structures. The first one is the Gray tensor product $\otimes$, sometimes called the lax Gray tensor product, first introduced by Al-Agl and Steiner [**1**], and then studied by Crans [**5**]. This tensor product generalizes the tensor product of 2-categories

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introduced by Gray in [**7**], hence its name. It is somehow a lax version of the cartesian product. For instance, one has

$$
\mathrm{D}_1 \otimes \mathrm{D}_1 = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} , \qquad
\mathrm{D}_1 \otimes \varDelta_2 = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} ,
$$

$$
\mathrm{D}_1 \otimes \mathrm{D}_2 = \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=A}] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] \arrow[d] \arrow[dr, phantom, "\Lleftarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left, dotted] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} ,
$$

where

$$
\mathrm{D}_1 = \bullet \longrightarrow \bullet , \quad \varDelta_2 = \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{and} \quad \mathrm{D}_2 = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} .
$$

In general, by iterating n times the Gray tensor product with D_1 starting from D_0, one gets a lax cube of dimension n. This (non-symmetric) tensor product defines a biclosed monoidal category structure and the two associated internal Hom are related to higher lax and oplax transformations. The second monoidal category structure is given by the join of ω-categories $\star$, introduced by the first author and Maltsiniotis in [**2**] to study slice ω-categories in a similar way as Joyal did for quasi-categories (see the introduction of [**2**] for more details). This operation, inspired by the topological join, is a higher dimensional lax version of the classical join of categories. For instance, one has

$$
\mathrm{D}_0 \star \mathrm{D}_0 = \bullet \longrightarrow \bullet = \mathrm{D}_1 \quad , \qquad
\mathrm{D}_0 \star \mathrm{D}_1 = \begin{tikzcd} & \bullet \arrow[dd] \\ \bullet \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Nearrow"] & {} \\ & \bullet \end{tikzcd} ,
$$

$$
\mathrm{D}_1 \star \mathrm{D}_1 = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \\ \bullet \arrow[r] & \bullet \arrow[u] \end{tikzcd}
\Rrightarrow
\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \arrow[r] \arrow[ur] & \bullet \arrow[u] \end{tikzcd} .
$$

More generally, by iterating n times the join with D_0 starting from D_0, one gets Street's n-th oriental $\mathcal{O}_n$ [**23**]. The join only admits "local internal Hom", in some appropriate sense, that are given by "generalized slice ω-categories". The Gray tensor product and the join are two fundamental structures of the theory of ω-categories.

The main goal of this paper is to prove that both the Gray tensor product and the join interact well with the folk model category structure or, more precisely, that they both define a monoidal model category structure on ω-$\mathcal{C}at$ endowed with the folk model category structure. Concretely, this means that if $i : X \to Y$ and $j : Z \to T$ are two folk cofibrations, then their pushout-product, that is, the ω-functor

$$i \otimes' j : Y \otimes Z \amalg_{X \otimes Z} X \otimes T \to Y \otimes T$$

induced by the commutative square

$$\begin{tikzcd} X \otimes Z \arrow[r, "X \otimes j"] \arrow[d, "i \otimes Z"'] & X \otimes T \arrow[d, "i \otimes T"] \\ Y \otimes Z \arrow[r, "Y \otimes j"'] & Y \otimes T \end{tikzcd} ,$$

is a folk cofibration, and a folk trivial cofibration if moreover either i or j is a folk trivial cofibration; and likewise for the join. This implies in particular that the Gray tensor product and the join can be left-derived as functors of two variables.

Note that the fact that the pushout-product, for the Gray tensor product, of two folk cofibrations is a folk cofibration was already proved by the second author in [**16**] by means of cubical ω-categories. Moreover, the particular case saying that the Gray tensor product of two cofibrant ω-categories is cofibrant was also established by Hadzihasanovic in [**8**]. Our proof, which is based on Steiner's theory of augmented directed complexes [**22**] and results of the first author and Maltsiniotis about pushouts of these [**2**, Chapter 3], is completely different and has the advantage to adapt easily to the case of the join. The hard part in showing the compatibility of the Gray tensor product and the join with the folk model category structure is then to prove that the pushout-product of a folk cofibration and a folk trivial cofibration is a folk trivial cofibration.

In the case of the Gray tensor product, we prove a more general result. Let (m, n)-$\mathcal{C}at$, for $0 \leqslant n \leqslant m \leqslant \omega$, be the category of (m, n)-categories, that is, the category of (strict) m-categories whose k-cells are strictly invertible as soon as $k > n$. Denote by $r : \omega\text{-}\mathcal{C}at \to (m, n)\text{-}\mathcal{C}at$ the left adjoint to the inclusion functor $(m, n)\text{-}\mathcal{C}at \hookrightarrow \omega\text{-}\mathcal{C}at$. It follows from [**15**] and [**3**] that the folk model category structure can be transferred along this adjunction to (m, n)-$\mathcal{C}at$. We prove, first, that the Gray tensor product induces, using r, a biclosed monoidal product for (m, n)-categories and, second, that this Gray tensor product of (m, n)-categories is compatible with the transferred model category structure on (m, n)-$\mathcal{C}at$. In particular, in the case $n = 0$, we get a monoidal model category structure on the category of (strict) m-groupoids. We prove that this structure is symmetric and satisfies the so-called monoid axiom of Schwede and Shipley [**21**]. This implies that the category of Gray monoids, that is, of monoid objects in the category of ω-groupoids endowed with the Gray tensor product, bears a canonical model category structure. This result was one of the motivating starting point of this work, as the second author showed in [**16**] that Gray monoids provide a good framework for higher dimensional rewriting.

On our way to show these results, we prove several properties of independent interest related to the Gray tensor product:

- We prove that if x is an m-cell of an ω-category X and y is an n-cell of an ω-category Y, then the associated $(m+n)$-cell $x \otimes y$ is reversible (that is, weakly invertible) if either x or y is reversible.
- We show that the analogous statement for strictly invertible cells holds. This implies that the tensor product of two ω-groupoids is an ω-groupoid.
- We prove that if X is a cofibrant ω-category, then $\mathrm{J}_1 \otimes X$, where J_1 is the ω-category obtained by factorizing the codiagonal of the terminal ω-category into a folk cofibration followed by a folk trivial fibration, is a cylinder object for X in the folk model category structure.
- We show that the invertible cells of the ω-category $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$, defined by the adjunction

$$\mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(T \otimes X, Y) \simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(T, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)),$$

  are precisely the component-wise invertible higher oplax transformations. This implies that if Y is an (m, n)-category, then so is $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$.
- We construct an ω-functor

$$X \otimes (Y \star Z) \to (X \otimes Y) \star Z,$$

  natural in X, Y and Z in ω-$\mathcal{C}at$, defining a tensorial strength on the functor $- \star Z$ for the Gray tensor product.

Finally, in an appendix, we prove that the "local internal $\underline{\mathsf{Hom}}$" of the join, the so-called generalized slices, can be right-derived as functors of two variables. By "local internal $\underline{\mathsf{Hom}}$", we mean the right adjoints of the functors

$$\begin{aligned} \omega\text{-}\mathcal{C}at &\to {}_{X}\backslash \omega\text{-}\mathcal{C}at \\ Y &\mapsto (X \star Y, X \hookrightarrow X \star Y) \end{aligned} \qquad \begin{aligned} \omega\text{-}\mathcal{C}at &\to {}_{Y}\backslash \omega\text{-}\mathcal{C}at \\ X &\mapsto (X \star Y, Y \hookrightarrow X \star Y), \end{aligned}$$

by opposition to the right adjoints of the functors

$$\begin{aligned} \omega\text{-}\mathcal{C}at &\to \omega\text{-}\mathcal{C}at \\ Y &\mapsto X \star Y \end{aligned} \qquad \begin{aligned} \omega\text{-}\mathcal{C}at &\to \omega\text{-}\mathcal{C}at \\ X &\mapsto X \star Y, \end{aligned}$$

which do not exist in the case of the join. These two local internal $\underline{\mathsf{Hom}}$, like classical internal $\underline{\mathsf{Hom}}$, can be promoted to functors of two variables, but in the local case, we get functors from the twisted arrow category:

$$\begin{aligned} \mathrm{Tw}(\omega\text{-}\mathcal{C}at) &\to \omega\text{-}\mathcal{C}at \\ X \xrightarrow{u} Z &\mapsto {}_{u}\backslash Z \end{aligned} \qquad \begin{aligned} \mathrm{Tw}(\omega\text{-}\mathcal{C}at) &\to \omega\text{-}\mathcal{C}at \\ Y \xrightarrow{v} Z &\mapsto Z \overset{\mathrm{co}}{/}_{v}. \end{aligned}$$

We prove, in the general setting of locally biclosed monoidal category introduced in [**2**], that these functors can be right-derived. This requires the use the theory of right simplicial derivability structures of Kahn and Maltsiniotis [**12**] as the twisted arrow category of a complete and cocomplete category is neither finitely cocomplete nor finitely complete in general.

We were unable to answer the following obvious question: is the tensor product of two folk weak equivalences a folk weak equivalence? Of course, a similar question can be asked for the join. We leave these two questions as open problems.

Our paper is organized as follows. In the first section, we recall the definitions related to the folk model category structure on the category $\omega\text{-}\mathcal{C}at$ of (strict) ω-categories. In particular, we define reversible cells (that is, weakly invertible cells). Using the Gray tensor product, whose definition is recalled in the next section, we introduce oplax transformations and reversible transformations. We recall the definition of the ω-category of cylinders and we end the section by introducing some classical dualities of $\omega\text{-}\mathcal{C}at$.

The purpose of the second section is to recall the definition of the Gray tensor product. We start by a summary of Steiner's theory of augmented directed complexes [**22**] and we use this theory to introduce, following [**22**] and [**2**], the Gray tensor product and its associated internal $\underline{\mathsf{Hom}}$, namely $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}$.

The aim of the third section is to prove that the pushout-product, for the Gray tensor product, of two folk cofibrations is a folk cofibration. We start by recalling the notion of a rigid monomorphism of augmented directed complexes with basis and some results from [**2**] of compatibility between pushouts of augmented directed complexes and pushouts of ω-categories. We then prove that the pushout-product, for the tensor product of augmented directed complexes, of two rigid monomorphisms is a rigid monomorphism. We then deduce the analogous result for ω-categories and folk cofibrations.

In the fourth section, we prove that if X is a folk cofibrant ω-category, then $\mathrm{J}_1 \otimes X$, where J_1 is the ω-category obtained by factorizing the codiagonal of the terminal ω-category into a cofibration followed by a trivial fibration, is a cylinder object for X in the folk model category. On our way to do so, we prove that the tensor product of a reversible (resp. invertible) m-cell by any other n-cell is reversible (resp. invertible). We start by proving the case $m = 1$ providing explicit formulas and then prove the general case by induction.

In the fifth section, we end the proof of the fact that the Gray tensor product makes of $\omega\text{-}\mathcal{C}at$ endowed with the folk model category structure a monoidal model category. Our strategy is abstracted in a general lemma whose main hypothesis, besides the fact that the pushout-product of two generating cofibrations is a cofibration, is the fact that the tensor product of a generating trivial cofibration and an object is a weak equivalence. We prove this hypothesis for the Gray tensor product using results from the previous section. We then prove some additional properties of the resulting monoidal model category.

In the sixth section, we introduce the category (m,n)-$\mathcal{C}at$ of (strict) (m,n)-categories and we study the interactions between the Gray tensor product and these (m,n)-categories. We prove that the invertible cells of the ω-category $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X,Y)$, defined by the adjunction $\mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(T \otimes X, Y) \simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(T, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(X,Y))$, are precisely the component-wise invertible higher oplax transformations. As a consequence, we obtain that if Y is an (m,n)-category, then so is $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X,Y)$. This implies by a result of Day [**6**] that the Gray tensor product induces, using the reflection functor $r : \omega\text{-}\mathcal{C}at \to (m,n)\text{-}\mathcal{C}at$, a biclosed monoidal category structure on (m,n)-$\mathcal{C}at$. In the case of m-groupoids (that is, the case $n = 0$), we show that the resulting monoidal product is symmetric. We introduce the folk model category structure on (m,n)-$\mathcal{C}at$, that is, the model category structure obtained by transferring along r the folk model category structure on ω-$\mathcal{C}at$, and we prove that the monoidal product on (m,n)-$\mathcal{C}at$ induced by the Gray tensor product is compatible with this structure. In the case of m-groupoids, we prove that the resulting monoidal model category satisfies the monoid axiom of Schwede and Shipley [**21**]. As a consequence, by results of Harper [**9**] and Muro [**19**], we obtain model category structures on the categories of algebras in ω-$\mathcal{G}pd$ over a given non-symmetric operad in ω-$\mathcal{G}pd$.

In the seventh section, we recall the definition of the join of ω-categories, introduced in [**2**], and its associated local internal $\underline{\mathsf{Hom}}$, the generalized slices. We prove that the join makes of ω-$\mathcal{C}at$ endowed with the folk model category structure a monoidal model category. The proof, that we only sketch, is very similar to the one for the Gray tensor product, and only requires one additional tool: the existence of an ω-functor $X \otimes (Y \star Z) \to (X \otimes Y) \star Z$ defining a tensorial strength on the functor $- \star Z$ for the Gray tensor product.

Finally, in an appendix, we recall the definition of a monoidal model category and how in this setting the monoidal tensor and, in the biclosed setting, the associated internal $\underline{\mathsf{Hom}}$ can be derived as functors of two variables. We then adapt this last result to the case of locally biclosed monoidal products, introduced in [**2**], our example of interest being the join of ω-categories. More precisely, we prove that the local internal $\underline{\mathsf{Hom}}$ of such a monoidal product can be right-derived as functors from the twisted arrow category. To do so, we endow the twisted arrow category of a model category with a right simplicial derivability structure in the sense of Kahn and Maltsiniotis [**12**], proving that right simplicial derivability structures can be lifted along discrete opfibrations.

1. Preliminaries on the folk model category structure

We will now describe the so-called "folk" model category structure on ω-$\mathcal{C}at$ introduced by Lafont, Métayer and Worytkiewicz in [**15**]. We start by describing the weak equivalences of this structure: the equivalences of ω-categories.

1.1. — We will denote by $\omega\text{-}\mathcal{C}at$ the category of strict ω-categories and strict ω-functors. As all the ω-categories and ω-functors in this paper will be strict, we will drop the adjective "strict" from now on. We will say that a cell of an ω-category is *trivial* if it is the identity on a cell of lower dimension. If x is an n-cell of an ω-category with $n \geqslant 1$, we will denote by 1_x the identity on x, by sx its source $(n-1)$-cell and by tx its target $(n-1)$-cell. If x is an n-cell with $n \geqslant 0$, for $0 \leqslant k \leqslant n$, we will denote by $s_k(x)$ its iterated source k-cell and by $t_k(x)$ its iterated target k-cell.

1.2. — Let X be an ω-category. By a *structure of reversibility* on X, we mean a set R of cells of X such that, if $u : x \to y$ is an n-cell in R, then there exists an n-cell $\bar{u} : y \to x$ and $(n+1)$-cells $\bar{r} *_n r \to 1_x$ and $r *_n \bar{r} \to 1_y$ all three in R. We say that an n-cell u of X is *reversible* if $n \geqslant 1$ and if there exists a structure of reversibility R on X containing u. A cell $\bar{u}$ in R as in the definition of a structure of reversibility is then called *a reverse* of u.

If C is a set of cells of X, to prove that every cell of C is reversible, it suffices to produce, for every n-cell u of C, a formula giving a reverse of u *assuming* that the $(n+1)$-cells of C are reversible. Indeed, one can then consider the subcategory R of X generated by the reversible cells of X, the cells in C and the cells given by the formulas, and show that the cells of R form a structure of reversibility. This is sometimes called *reasoning by coinduction.*

1.3. — An ω-functor $f : X \to Y$ is an *equivalence of ω-categories* or *folk weak equivalence* if:

- for every 0-cell y of Y, there exists a 0-cell x of X and a reversible 1-cell $f(x) \to y$ of Y,
- for every $n \geqslant 1$, every pair of parallel $(n-1)$-cells x, x' of X and every n-cell $v : f(x) \to f(x')$ of Y, there exists an n-cell $u : x \to x'$ of X and a reversible $(n+1)$-cell $f(u) \to v$ of Y.

We now move on to the description of generating cofibrations and trivial cofibrations of the folk model category structure.

1.4. — For every $n \geqslant 0$, we will denote by D_n the free-standing n-cell in $\omega\text{-}\mathcal{C}at$. In other words, the ω-category D_n corepresents the functor sending an ω-category to its set of n-cells. This ω-category D_n is actually an n-category. It has a unique non-trivial n-cell that we will call its *principal cell.* Here are pictures of D_n for small n:

$$\mathrm{D}_0 = \{0\}\,, \quad \mathrm{D}_1 = \begin{tikzcd} 0 \arrow[r] & 1 \end{tikzcd}\,, \quad \mathrm{D}_2 = \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd} \quad \text{and} \quad \mathrm{D}_3 = \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \Rrightarrow \Downarrow"] & 1 \end{tikzcd}\,.$$

If x is an n-cell of an ω-category X, we will denote by $\langle x \rangle : \mathrm{D}_n \to X$ the corresponding ω-functor.

1.5. — Let $n \geqslant 0$. We will denote by $\partial\mathrm{D}_n$ the $(n-1)$-category obtained from D_n by removing its principal cell. In other words, $\partial\mathrm{D}_0$ is the empty ω-category (which is a (-1)-category!) and, for $n \geqslant 1$, $\partial\mathrm{D}_n$ is the free-standing pair of parallel $(n-1)$-cells in ω-$\mathcal{C}at$. Here are pictures of $\partial\mathrm{D}_n$ for small n:

$$\partial\mathrm{D}_0 = \{\,\}\,, \quad \partial\mathrm{D}_1 = \{\,0 \quad 1\,\}\,, \quad \partial\mathrm{D}_2 = \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] & 1 \end{tikzcd} \quad \text{and} \quad \partial\mathrm{D}_3 = \begin{tikzcd} 0 \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow\quad\Downarrow"] & 1 \end{tikzcd}\quad .$$

If $n \geqslant 1$ and x, y are two parallel $(n-1)$-cells of an ω-category X, we will denote by $\langle x, y\rangle : \partial\mathrm{D}_n \to X$ the corresponding ω-functor.

For every $n \geqslant 0$, we have a canonical inclusion

$$i_n : \partial\mathrm{D}_n \hookrightarrow \mathrm{D}_n,$$

and, for $n \geqslant 1$, two ω-functors

$$s, t : \mathrm{D}_{n-1} \to \partial\mathrm{D}_n$$

corresponding to the source and target of the principal cell of D_n, respectively.

1.6. — We will denote by I the set

$$I = \{i_n : \partial\mathrm{D}_n \hookrightarrow \mathrm{D}_n \mid n \geqslant 0\}.$$

As the category ω-$\mathcal{C}at$ is locally presentable, this set generates a weak factorization system on ω-$\mathcal{C}at$. The ω-functors in the left class (that is, the retracts of transfinite compositions of pushouts of elements of I) will be called *folk cofibrations* or simply *cofibrations*; as for the ω-functors in the right class (that is, the ω-functors having the right lifting property with respect to I), they will be called *folk trivial fibrations* or simply *trivial fibrations*.

1.7. — Let $n \geqslant 1$. Consider the ω-functor $\langle d, d\rangle : \partial\mathrm{D}_n \hookrightarrow \mathrm{D}_{n-1}$, where d denotes the principal cell of D_{n-1}. Fix a factorization

$$\partial\mathrm{D}_n \xrightarrow{k_n} \mathrm{J}_n \xrightarrow{q_n} \mathrm{D}_{n-1}$$

of this ω-functor into a folk cofibration k_n followed by a folk trivial fibration q_n. As i_n is a folk cofibration and q_n is a folk trivial fibration, the commutative square

$$\begin{tikzcd} \partial\mathrm{D}_n \arrow[r, "k_n"] \arrow[d, "i_n"'] & J_n \arrow[d, "q_n"] \\ \mathrm{D}_n \arrow[r, "\langle 1_d\rangle"'] \arrow[ur, dotted] & \mathrm{D}_{n-1} \end{tikzcd}\quad ,$$

where d still denotes the principal cell of D_{n-1}, admits a lift. We fix such a lift

$$l_n : \mathrm{D}_n \to \mathrm{J}_n\,.$$

By definition, the *principal cell of* J_n is the image of the principal cell of D_n by l_n.

We will denote by
$$j_n : \mathrm{D}_{n-1} \to \mathrm{J}_n$$
the composite
$$\mathrm{D}_{n-1} \xrightarrow{s} \partial\mathrm{D}_n \xrightarrow{k_n} \mathrm{J}_n,$$
that is, the ω-functor corresponding to the source of the principal cell of J_n, and by J the set
$$J = \{j_n : \mathrm{D}_{n-1} \to \mathrm{J}_n \mid n \geqslant 1\}.$$

Theorem 1.8 **(Lafont–Métayer–Worytkiewicz)**. — *The category ω-$\mathcal{C}at$ is endowed with a model category structure, cofibrantly generated by I and J, whose weak equivalences are the folk weak equivalences and whose cofibrations are the folk cofibrations. All the ω-categories are fibrant for this model category structure.*

Proof. — This is [**15**, Theorem 4.39 and Proposition 5.1]. □

The model category structure of the previous theorem is known as the *folk model category structure* on ω-$\mathcal{C}at$. We will now describe a path object for this structure. We start by some preliminaries on oplax transformations.

1.9. — If X and Y are two ω-categories, we will denote by $X \otimes Y$ their Gray tensor product. We refer the reader to Section 2 for more details and a precise definition of this tensor product, based on Steiner's work [**22**]. Let us only recall that the Gray tensor product defines a (non-symmetric) biclosed monoidal category structure whose unit is the terminal ω-category D_0. Its right and left internal Hom will be denoted by $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}$, respectively, so that if X, Y and Z are three ω-categories, we have natural bijections
$$\mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(Y, Z)) \simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X \otimes Y, Z) \simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(Y, \underline{\mathsf{Hom}}_{\mathrm{lax}}(X, Z)).$$

1.10. — Let X and Y be two ω-categories. By adjunction, the set of 0-cells of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$ can be identified with the set of ω-functors $\mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X, Y)$. If $f, g : X \to Y$ are two ω-functors, an *oplax transformation* $\alpha : f \Rightarrow g$ is a 1-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$ from f to g. Such an oplax transformation can be identified with a functor
$$h : \mathrm{D}_1 \otimes X \to Y$$
making the diagram
$$\begin{tikzcd}
X \arrow[dr, "\langle 0\rangle\otimes X"'] \arrow[drr, bend left, "f"] & & \\
& \mathrm{D}_1 \otimes X \arrow[r, "h"] & Y \\
X \arrow[ur, "\langle 1\rangle\otimes X"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd},$$

where X is identified with $\mathrm{D}_0 \otimes X$, commute. Alternatively, again by adjunction, such an oplax transformation can be seen as an ω-functor

$$k : X \to \Gamma(Y),$$

where $\Gamma(Y) = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, Y)$, making the diagram

$$\begin{tikzcd} & & Y \\ X \arrow[rru, bend left, "f"] \arrow[r, "k"] \arrow[rrd, bend right, "g"'] & \Gamma(Y) \arrow[ru, "\pi^-"'] \arrow[rd, "\pi^+"] & \\ & & Y \end{tikzcd} ,$$

where Y is identified with $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, Y)$ and

$$\pi^- = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\langle 0\rangle, Y) \quad \text{and} \quad \pi^+ = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\langle 1\rangle, Y),$$

commute.

One can define lax transformations in a similar way.

1.11. — Let x be an m-cell of an ω-category X and let y be an n-cell of an ω-category Y. One defines an $(m+n)$-cell $x\otimes y$ of $X\otimes Y$ in the following way. The ω-category $\mathrm{D}_m \otimes \mathrm{D}_n$ is an $(m+n)$-category that admits a unique non-trivial $(m+n)$-cell. We will call this cell the *principal cell* of $\mathrm{D}_m \otimes \mathrm{D}_n$. The $(m+n)$-cell $x \otimes y$ is the cell corresponding to the ω-functor

$$\mathrm{D}_{m+n} \xrightarrow{\langle p\rangle} \mathrm{D}_m \otimes \mathrm{D}_n \xrightarrow{\langle x\rangle\otimes\langle y\rangle} X \otimes Y,$$

where p denotes the principal cell of $\mathrm{D}_m \otimes \mathrm{D}_n$.

1.12. — Let $f, g : X \to Y$ be two ω-functors and let $\alpha : f \Rightarrow g$ be an oplax transformation. Denote by $h : \mathrm{D}_1 \otimes X \to Y$ the corresponding ω-functor and by (01) the principal cell of D_1. If x is an n-cell of X, the *component* of α at x is the $(n+1)$-cell of Y

$$\alpha_x = h((01) \otimes x).$$

As the ω-category $\mathrm{D}_1\otimes X$ is generated by cells of the form $0\otimes x$, $1\otimes x$ and $(01)\otimes x$, with x a cell of X, the transformation α is entirely determined by its components. Furthermore, oplax transformations can be defined purely in terms of their components (see [**2**, paragraph 1.9 and Section B.2]).

1.13. — If Y is an ω-category, the n-cells of $\Gamma(Y)$ are called *n-cylinders* in Y. By adjunction, they correspond to ω-functors $c : \mathrm{D}_1 \otimes \mathrm{D}_n \to Y$. If c is such an n-cylinder, we can set

$$x = c(0 \otimes d) \quad \text{and} \quad y = c(1 \otimes d)$$

and, for $0 \leqslant k \leqslant n$,

$$\alpha_k^- = c((01) \otimes s_k(d)) \quad \text{and} \quad \alpha_k^+ = c((01) \otimes t_k(d)),$$

where (01) denotes the principal cell of D_1 and d the one of D_n. Note that $\alpha_n^- = \alpha_n^+$ and we will often write α_n for this cell. These cells completely determine c and we will often write $c = (x, y, \alpha)$. Moreover, by [**2**, Proposition B.1.6], n-cells x and y and $(k+1)$-cells α_k^-, α_k^+, for $0 \leqslant k \leqslant n$, with $\alpha_n^- = \alpha_n^+$, determine an n-cylinder if and only if one has

$$\alpha_k^\varepsilon : \alpha_{k-1}^+ *_{k-1} \alpha_{k-2}^+ *_{k-2} \cdots *_1 \alpha_0^+ *_0 x_k^\varepsilon \to y_k^\varepsilon *_0 \alpha_0^- *_1 \cdots *_{k-1} \alpha_{k-1}^-,$$

for $\varepsilon = \pm$, where $x_k^- = s_k(x)$ and $x_k^+ = t_k(x)$, and similarly for y.

If $c = (x, y, \alpha)$ is an n-cylinder, the cell $\alpha_n^- = \alpha_n^+$ is called the *principal cell* of c. We say that c is *reversible* if all the cells α_k^ε for $0 \leqslant k \leqslant n$ and $\varepsilon = \pm$ are reversible. It follows from the explicit formulas describing the operations of the ω-category $\Gamma(Y)$ (see [**15**, Appendix A] or [**2**, Proposition B.1.15]) that the graded subset $\Gamma_{\mathrm{rev}}(Y)$ of $\Gamma(Y)$ consisting of reversible cylinders is actually a sub-ω-category.

1.14. — Let $f, g : X \to Y$ be two ω-functors and let $\alpha : f \Rightarrow g$ be an oplax transformation. The transformation α is said to be *reversible* if, for every cell x of X, the component α_x is a reversible cell of Y. A reversible oplax transformation will be simply called a *reversible transformation.*

Essentially by definition, the transformation α is reversible if and only if the corresponding ω-functor $X \to \Gamma(Y)$ factors through the inclusion $\Gamma_{\mathrm{rev}}(Y) \hookrightarrow \Gamma(Y)$. In other words, the data of a reversible transformation $\alpha : f \Rightarrow g$ corresponds to the data of an ω-functor

$$k : X \to \Gamma_{\mathrm{rev}}(Y)$$

making the obvious diagram

$$\begin{tikzcd} & & Y \\ X \arrow[r, "k"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & \Gamma_{\mathrm{rev}}(Y) \arrow[ur, "\pi^-"'] \arrow[dr, "\pi^+"] & \\ & & Y \end{tikzcd}$$

commute.

1.15. — Let $f : X \to Y$ be an ω-functor. The identity on f seen as a 0-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$ defines an oplax transformation $1_f : f \Rightarrow f$. This transformation is easily seen to be reversible (its components are identities).

In particular, by applying this to the identity ω-functor $1_X : X \to X$, we get a commutative diagram

$$\begin{tikzcd} & & X \\ X \arrow[r, "\iota"] \arrow[rru, bend left, "1_X"] \arrow[rrd, bend right, "1_X"'] & \Gamma_{\mathrm{rev}}(X) \arrow[ru, "\pi^-"'] \arrow[rd, "\pi^+"] & \\ & & X \end{tikzcd}$$

or, in other words, a factorization

$$X \xrightarrow{\iota} \Gamma_{\mathrm{rev}}(X) \xrightarrow{\pi} X \times X$$

of the diagonal functor.

***Theorem 1.16* (Lafont–Métayer–Worytkiewicz).** — *For every ω-category X, the factorization*

$$X \xrightarrow{\iota} \Gamma_{\mathrm{rev}}(X) \xrightarrow{\pi} X \times X$$

of the diagonal is a path object for the folk model category structure, in the sense that ι a weak equivalence and that π is a fibration.

Proof. — This is [**15**, Proposition 4.45]. $\square$

Remark 1.17. — A right homotopy with respect to the path object of the previous theorem is precisely a reversible transformation.

We will now describe the cofibrant objects of the folk model category structure.

1.18. — Let X be an ω-category. For $m \geqslant -1$, we will denote by $X_{\leqslant m}$ the m-category obtained from X by removing the non-trivial k-cells for $k > m$. In particular, if $m = -1$, we have $X_{\leqslant -1} = \varnothing$. There is an obvious inclusion ω-functor $X_{\leqslant m} \hookrightarrow X_{\leqslant m+1}$.

Let B be a set of cells of X. We will say that X is *freely generated by B* if, for every $n \geqslant 0$, the commutative square

$$\begin{tikzcd} \coprod_{x \in B_n} \partial \mathrm{D}_n \arrow[r, "\langle sx, tx \rangle"] \arrow[d, "\coprod_{x \in B_n} i_n"'] & X_{\leqslant n-1} \arrow[d] \\ \coprod_{x \in B_n} \mathrm{D}_n \arrow[r, "\langle x \rangle"'] & X_{\leqslant n} \end{tikzcd} ,$$

where B_n denotes the set of n-cells in B and the right vertical arrow is the canonical inclusion, is a pushout square.

One says that an ω-category is *free in the sense of polygraphs* if it admits a set of cells that freely generates it.

***Theorem 1.19* (Métayer).** — *The cofibrant objects of the folk model category structure are the ω-categories that are free in the sense of polygraphs.*

Proof. — This is the main result of [**17**]. $\square$

We end the section by introducing important dualities of ω-$\mathcal{C}at$ and some of their properties.

1.20. — If X is an ω-category, we will denote by X^{op} (resp. by X^{co}) the ω-category obtained from X by reversing the direction of the cells of odd (resp. even) dimension. The assignments $X \mapsto X^{\mathrm{op}}$ and $X \mapsto X^{\mathrm{co}}$ are both involutive automorphisms of the category ω-$\mathcal{C}at$. Moreover, they are anti-monoidal in the sense that the assignment $x \otimes y \mapsto y \otimes x$ defines isomorphisms

$$(X \otimes Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes X^{\mathrm{op}} \quad \text{and} \quad (X \otimes Y)^{\mathrm{co}} \simeq Y^{\mathrm{co}} \otimes X^{\mathrm{co}}.$$

Furthermore, there are canonical isomorphisms

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)^{\mathrm{op}} \simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(X^{\mathrm{op}}, Y^{\mathrm{op}}),$$
$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)^{\mathrm{co}} \simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(X^{\mathrm{co}}, Y^{\mathrm{co}})$$

(see for instance [**2**, Propositions A.22 and A.23]).

The symmetry of the definition of a reversible cell shows that a cell is reversible in X if and only if the corresponding cell is reversible in X^{op} (resp. in X^{co}). This easily implies that an ω-functor $f : X \to Y$ is a folk weak equivalence if and only if $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ (resp. $f^{\mathrm{co}} : X^{\mathrm{co}} \to Y^{\mathrm{co}}$) is. Moreover, for every $n \geqslant 0$, the ω-functor i_n^{op} (resp. i_n^{co}) can be identified with the ω-functor $i_n : \partial\mathrm{D}_n \hookrightarrow \mathrm{D}_n$. This implies that an ω-functor i is a folk cofibration if and only if i^{op} (resp. i^{co}) is, and hence that j is a folk trivial cofibration if and only if j^{op} (resp. j^{co}) is.

2. Preliminaries on the Gray tensor product

The purpose of this section is to define the Gray tensor product of ω-categories. This tensor product was introduced by Al-Agl and Steiner [**1**] as a generalization of Gray's tensor product of 2-categories [**7**], and is somehow a lax version of the cartesian product. The definition we will give in this section is based on Steiner's theory of augmented directed complexes [**22**]. The strategy, due to Steiner, is the following. Steiner's complexes are a tool to describe a large subclass of the class of free ω-categories in the sense of polygraphs. The usual tensor product of chain complexes induces a tensor product on these free ω-categories. The general Gray tensor product is then obtained by density of this subclass in the category of ω-categories.

We start by briefly recalling Steiner's theory.

2.1. — An *augmented directed complex* is an augmented chain complex of abelian groups in nonnegative degree

$$\cdots \xrightarrow{d} K_n \xrightarrow{d} K_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} K_0 \xrightarrow{e} \mathbb{Z},$$

endowed with, for every $n \geqslant 0$, a submonoid K_n^* of K_n of so-called *positive elements.* If K and L and two augmented directed complexes, a *morphism* $f : K \to L$ is a morphism of the underlying augmented chain complexes respecting the positive elements, that is, such that, for every $n \geqslant 0$, we have $f(K_n^*) \subset L_n^*$. We will denote by $\mathcal{C}_{\mathrm{ad}}$ the category of augmented directed complexes.

2.2. — In [**22**], Steiner defines a functor

$$\nu : \mathcal{C}_{\mathrm{ad}} \to \omega\text{-}\mathcal{C}at.$$

We refer the reader to [**22**, Definition 1.6] (or [**2**, paragraph 2.4]) for a detailed definition. Let us just mention that if K is an augmented directed complex, then the n-cells of the ω-category $\nu(K)$ are given by tables

$$\begin{pmatrix} x_0^- & \cdots & x_n^- \\ x_0^+ & \cdots & x_n^+ \end{pmatrix},$$

where
- x_i^- and x_i^+ are in K_i^*, for $0 \leqslant i \leqslant n$,
- $x_n^- = x_n^+$,
- $d(x_i^-) = x_{i-1}^+ - x_{i-1}^- = d(x_i^+)$, for $0 < i \leqslant n$,
- $e(x_0^-) = 1$ and $e(x_0^+) = 1$.

2.3. — If K is an augmented directed complex, a *basis* of K is a graded set $(B_n)_{n\geqslant 0}$ such that, for every $n \geqslant 0$,
- B_n is a basis of the $\mathbb{Z}$-module K_n,
- B_n generates the submonoid K_n^*.

One shows that if such a basis exists, then it is unique.

2.4. — If K is an augmented directed complex with basis (B_n), then for every n-chain x, one can write $x = \sum_{b \in B_n} n_b b$, where the n_b are integers, in a unique way, and we set

$$x^- = \sum_{\substack{b \in B_n \\ n_b < 0}} (-n_b) b \quad \text{and} \quad x^+ = \sum_{\substack{b \in B_n \\ n_b > 0}} n_b b.$$

If x is an n-chain with $n > 0$, we set

$$d^- x = (dx)^- \quad \text{and} \quad d^+ x = (dx)^+.$$

2.5. — Let K be an augmented directed complex with basis. For every n-chain x in the basis, we define a table

$$\langle x\rangle = \begin{pmatrix} x_0^- & \cdots & x_n^- \\ x_0^+ & \cdots & x_n^+ \end{pmatrix},$$

by induction, setting

- $x_n^- = x$ and $x_n^+ = x$,
- $x_i^- = d^-(x_{i+1}^-)$ and $x_i^+ = d^+(x_{i+1}^+)$, for $0 \leqslant i < n$.

This table is an n-cell of $\nu(K)$ if and only if $e(x_0^-) = 1$ and $e(x_0^+) = 1$. In this case, one says that the n-cell $\langle x\rangle$ is the *atom* associated to x.

The augmented directed complex with basis K is said to be *unital* if, for every element x of the basis of K, one has $e(x_0^-) = 1$ and $e(x_0^+) = 1$.

2.6. — One says that an augmented directed complex K with basis (B_n) is *strongly loop-free* if there exists a partial order $\preceq$ on $\coprod_{n\geqslant 0} B_n$ such that, for every $n > 0$, every x in B_n, and every y and z in the support (according to the basis B_{n-1}) of d^-x and d^+x, respectively, one has

$$y \preceq x \preceq z.$$

2.7. — A *strong Steiner complex* is an augmented directed complex with basis that is both unital and strongly loop-free. We will denote by $\mathcal{S}t_{\mathrm{f}}$ the full subcategory of $\mathcal{C}_{\mathrm{ad}}$ consisting of Steiner complexes.

***Theorem 2.8* (Steiner)**. — *The functor $\nu_{|\mathcal{S}t_{\mathrm{f}}} : \mathcal{S}t_{\mathrm{f}} \to \omega\text{-}\mathcal{C}at$ is fully faithful. Moreover, if K is a strong Steiner complex, then $\nu(K)$ is freely generated (see paragraph 1.18) by its atoms.*

Proof. — This follows from [**22**, Proposition 3.7, Theorem 5.6 and Theorem 6.1]. $\square$

We will now define the Gray tensor product of ω-categories, starting with the tensor product of augmented directed complexes.

2.9. — The *tensor product* $K \otimes L$ of two augmented directed complexes K and L is defined in the following way:

- The underlying augmented complex of $K \otimes L$ is the usual one:
  - for $n \geqslant 0$, we have
  $$(K \otimes L)_n = \bigoplus_{i+j=n} K_i \otimes L_j,$$
  - for x in K_i and y in K_j, we have
  $$d(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy,$$
  where by convention $dz = 0$ if the degree of z is 0,
  - for x in K_0 and y in L_0, we have
  $$e(x \otimes y) = e(x)e(y).$$

– The submonoid $(K \otimes L)^*_n$ is defined to be generated by the subset

$$\bigoplus_{i+j=n} K^*_i \otimes L^*_j$$

of $(K \otimes L)_n$.

The tensor product defines a (non-symmetric) monoidal category structure on the category of augmented directed complexes. Its unit, that we will denote by $\mathbb{Z}$, is the complex concentrated in degree 0 of value $\mathbb{Z}$ with the identity augmentation and $\mathbb{N}$ as the submonoid of positive elements of degree 0. Steiner proved (see [**22**, Example 3.10]) that this monoidal category structure restricts to the full subcategory of strong Steiner complexes.

***Theorem 2.10* (Steiner, Ara–Maltsiniotis).** — *There exists a unique, up to unique isomorphism, biclosed monoidal category structure on ω-$\mathcal{C}at$ making the functor $\nu_{|\mathcal{S}t_{\mathrm{f}}} : \mathcal{S}t_{\mathrm{f}} \to \omega$-$\mathcal{C}at$ a monoidal functor, where $\mathcal{S}t_{\mathrm{f}}$ is endowed with the monoidal category structure given by the tensor product.*

Proof. — See [**22**, Section 7], whose proof was completed by [**2**, Theorem A.15]. ☐

2.11. — We define the *Gray tensor product* to be the tensor product given by the previous theorem. If X are Y are two ω-categories, their Gray tensor product will be denoted by $X \otimes Y$. Explicitly, one has

$$X \otimes Y = \varinjlim_{\substack{\nu(K) \to X,\, K \in \mathcal{S}t_{\mathrm{f}} \\ \nu(L) \to Y,\, L \in \mathcal{S}t_{\mathrm{f}}}} \nu(K \otimes L).$$

The unit of the Gray tensor product is the terminal ω-category D_0.

The right and left internal Hom of the Gray tensor product will be denoted by $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}$, respectively, so that if X, Y and Z are three ω-categories, we have natural bijections

$$\mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(Y,Z)) \simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X \otimes Y, Z) \simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(Y, \underline{\mathsf{Hom}}_{\mathrm{lax}}(X,Z)).$$

Examples 2.12. — Here are some examples of Gray tensor products of ω-categories:

$$\mathrm{D}_1 \otimes \mathrm{D}_1 = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}, \qquad \mathrm{D}_1 \otimes \Delta_2 = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd},$$

$$\mathrm{D}_1 \otimes \mathrm{D}_2 = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] \arrow[d] \arrow[dr, phantom, "\Lleftarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left, dotted] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd},$$

where $\Delta_2 = \bullet \longrightarrow \bullet \longrightarrow \bullet$.

Remark 2.13. — If x is an m-cell of an ω-category X and y is an n-cell of an ω-category Y, we saw in paragraph 1.11 that one can define an $(m+n)$-cell $x \otimes y$ of $X \otimes Y$. For instance, the tensor product of the principal cells of the disks appearing in the examples above is, in both cases, the unique non-trivial cell of maximal dimension. The formula that we gave as a definition for the Gray tensor product easily implies that the ω-category $X \otimes Y$ is generated by the set of cells of the form $x \otimes y$, with x a cell of X and y a cell of Y.

Remark 2.14. — The Gray tensor product used in this paper is what we like to call the *oplax* Gray tensor product. The *lax* version is the functor $(X, Y) \mapsto Y \otimes X$ and is actually the one introduced by Gray in the 2-categorical case [**7**]. The natural isomorphism $(X \otimes Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes X^{\mathrm{op}}$ and the stability of the data of the folk model category structure by the duality $Z \mapsto Z^{\mathrm{op}}$ (see paragraph 1.20) show that the results we prove in this paper for the oplax version of the Gray tensor product can be adapted to the lax version.

3. Compatibility of the tensor product with cofibrations

The purpose of this section is to prove that ω-$\mathcal{C}at$ endowed with the Gray tensor product $\otimes$ satisfies the part of the axioms of monoidal model categories (see paragraph A.1) dealing with cofibrations. In other words, given two folk cofibrations

$$i : X \to Y \quad \text{and} \quad j : Z \to T,$$

we will prove that the ω-functor

$$i \otimes' j : Y \otimes Z \amalg_{X \otimes Z} X \otimes T \to Y \otimes T,$$

is also a folk cofibration. This immediately follow from the case of generating cofibrations, for which we will use Steiner's theory.

We start by some supplements on pushouts of strong Steiner complexes.

3.1. — If K is an augmented directed complex with basis, we will denote its basis by B_K.

Let $f : K \to L$ be a monomorphism of augmented directed complexes with basis. One says that f is a *rigid monomorphism* if it sends elements of the basis B_K of K to elements of the basis B_L of L.

Proposition 3.2 **(Ara–Maltsiniotis)**. — *Consider a pushout square*

$$\begin{array}{ccc} K & \xrightarrow{u} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ L & \xrightarrow[v]{} & N \end{array}$$

in the category of augmented directed complexes such that:

– *K, L, M, N are strong Steiner complexes,*
– *f and u are rigid monomorphisms.*

Then
– *we have $B_N = B_L \amalg_{B_K} B_M$ (as sets),*
– *the morphisms g and v are rigid monomorphisms,*
– *the functor $\nu : \mathcal{C}_{\mathrm{ad}} \to \omega\text{-}\mathcal{C}at$ sends this square to a pushout square in $\omega\text{-}\mathcal{C}at$.*

Proof. — The first assertion is a particular case of [**2**, Proposition 3.6]. The second one follows from [**2**, Proposition 3.12]. As for the third one, it is a special case of [**2**, Theorem 3.8]. □

Remark 3.3. — The proposition remains true if one only assumes that the complexes are Steiner complexes (named "augmented directed complexes with a loop-free unital basis" in [**1**]) as opposed to strong Steiner complexes. We stated the more restrictive result only because we did not include the definition of a Steiner complex in this paper.

3.4. — If K and L are two augmented directed complexes with basis, one immediately checks that $K \otimes L$ is an augmented directed complex with basis

$$B_{K\otimes L} = B_K \otimes B_L = \{x \otimes y \mid x \in B_K,\, y \in B_L\}.$$

Proposition 3.5. — *Let $i : K \to L$ and $j : M \to N$ be two rigid monomorphisms between augmented directed complexes with basis. Then the morphism*

$$i \otimes' j : L \otimes M \amalg_{K\otimes M} K \otimes N \to L \otimes N$$

is a rigid monomorphism between augmented directed complexes with basis which identifies $L \otimes M \amalg_{K\otimes M} K \otimes N$ with the subcomplex generated by $B_L \otimes B_M \cup B_K \otimes B_N$.

Proof. — Colimits in the category of augmented directed complexes are computed degreewise (see [**2**, paragraph 3.1]). Let $n \geqslant 0$. If B is the basis of an augmented directed complex, we will denote by B_n the set of n-chains in B. As the free abelian group functor commutes with colimits, the abelian group $(L \otimes M \amalg_{K\otimes M} K \otimes N)_n$ is free with basis $(B_L \otimes B_M)_n \amalg_{(B_K \otimes B_M)_n} (B_K \otimes B_N)_n = (B_L \otimes B_M)_n \cup (B_K \otimes B_N)_n$. Similarly, the submonoid $(L \otimes M \amalg_{K\otimes M} K \otimes N)^*_n$ is generated by this basis. This proves that $L \otimes M \amalg_{K\otimes M} K \otimes N$ is free with basis $B_L \otimes B_M \cup B_K \otimes B_N$. Moreover, this shows that the map $(i \otimes' j)_n : (L \otimes M \amalg_{K\otimes M} K \otimes N)_n \to (L \otimes N)_n$ can be identified with the image of the map $(B_L \otimes B_M)_n \cup (B_K \otimes B_N)_n \hookrightarrow (B_L \otimes B_N)_n$ by the free abelian group functor. As this functor preserves monomorphisms, this implies that $i \otimes' j$ is a monomorphism. The fact that it is rigid being obvious, this ends the proof. □

Proposition 3.6. — *Let $i : K \to L$ and $j : M \to N$ be two rigid monomorphisms between strong Steiner complexes. Then the pushout square associated to*

$$L \otimes M \xleftarrow{i \otimes M} K \otimes M \xrightarrow{K \otimes j} K \otimes N$$

satisfies the hypotheses of Proposition 3.2.

Proof. — Strong Steiner complexes and rigid monomorphisms are both stable under tensor product by [**22**, Example 3.10] and [**2**, Proposition A.6]. It thus suffices to prove that $L \otimes M \amalg_{K \otimes M} K \otimes N$ is a strong Steiner complex. This follows immediately from the fact that, by the previous proposition, $L \otimes M \amalg_{K \otimes M} K \otimes N$ is a subcomplex of the strong Steiner complex $L \otimes N$ generated by a subset of its basis. $\square$

Proposition 3.7. — *Let $i : K \to L$ and $j : M \to N$ be two rigid monomorphisms between strong Steiner complexes. Then the ω-functor*

$$\nu(i) \otimes' \nu(j) : \nu(L) \otimes \nu(M) \amalg_{\nu(K) \otimes \nu(M)} \nu(K) \otimes \nu(N) \to \nu(L) \otimes \nu(N)$$

is a folk cofibration.

Proof. — By applying Proposition 3.2 to the pushout square of the previous proposition and using the fact that the functor $\nu_{|\mathcal{S}t_\mathrm{f}} : \mathcal{S}t_\mathrm{f} \to \omega\text{-}\mathcal{C}at$ is monoidal for the tensor product (Theorem 2.10), one gets that the ω-functor

$$\nu(i) \otimes' \nu(j) : \nu(L) \otimes \nu(M) \amalg_{\nu(K) \otimes \nu(M)} \nu(K) \otimes \nu(N) \to \nu(L) \otimes \nu(N)$$

can be identified with

$$\nu(i \otimes' j) : \nu(L \otimes M \amalg_{K \otimes M} K \otimes N) \to \nu(L \otimes N).$$

As the functor ν respects monomorphisms (this follows from its concrete description but also from the fact that it admits a left adjoint, see [**22**, Theorem 2.11]), $\nu(L \otimes M \amalg_{K \otimes M} K \otimes N)$ can be identified with a sub-ω-category of $\nu(L \otimes M)$. Moreover, by Steiner's Theorem 2.8, these two ω-categories are freely generated by their atoms, which, by Proposition 3.5, are in bijection with $B_L \otimes B_M \cup B_K \otimes B_N$ and $B_L \otimes B_N$, respectively. The ω-category $\nu(L \otimes N)$ can thus be obtained from $\nu(L \otimes M \amalg_{K \otimes M} K \otimes N)$ by freely adding cells (in the sense of taking pushouts along some $i_n : \partial\mathrm{D}_n \hookrightarrow \mathrm{D}_n$) indexed by $(B_L \otimes B_N) \backslash (B_L \otimes B_M \cup B_K \otimes B_N)$. The inclusion morphism is therefore a cofibration. $\square$

To apply the previous proposition to the generating cofibrations, we need the following lemma:

Lemma 3.8. — *Let $n \geqslant 0$. The ω-functor $i_n : \partial\mathrm{D}_n \hookrightarrow \mathrm{D}_n$ can be written $\nu(\lambda(i_n))$, where $\lambda(i_n) : \lambda(\partial\mathrm{D}_n) \hookrightarrow \lambda(\mathrm{D}_n)$ is a rigid monomorphism between strong Steiner complexes.*

Proof. — Let $\lambda(\mathrm{D}_n)$ be the free augmented directed complex with basis the set of non-trivial cells of D_n (see [**2**, paragraph 4.10] for an explicit description), let $\lambda(\partial\mathrm{D}_n)$ be the augmented directed subcomplex generated by the subset of non-trivial cells of $\partial\mathrm{D}_n$ and let $\lambda(i_n)$ be the resulting inclusion morphism. One easily checks that i_n can be identified with $\nu(\lambda(i_n))$, that $\lambda(\mathrm{D}_n)$ is indeed a strong Steiner complex (see [**22**, Example 4.7] or [**2**, paragraph 4.10]) and thus that any subset of its basis defines a strong Steiner subcomplex whose inclusion morphism is a rigid monomorphism. □

Theorem 3.9. — *If*

$$i : X \to Y \quad \textit{and} \quad j : Z \to T$$

are two folk cofibrations, then the ω-functor

$$i \otimes' j : Y \otimes Z \amalg_{X \otimes Z} X \otimes T \to Y \otimes T$$

is also a folk cofibration.

Proof. — By the classical Lemma A.3, it suffices to prove the result when i and j are generating cofibrations. But this case follows from Proposition 3.7 by the previous lemma. □

Remark 3.10. — The previous result was first established by the second author (see [**16**, Proposition 5.1.2.7]) using cubical sets. The advantage of the method of the present paper is that it will adapt directly to the join of ω-categories (see Section 7).

Corollary 3.11. — *The tensor product of two cofibrant ω-categories is a cofibrant ω-category.*

Proof. — Let X and Y be two ω-categories. The corollary follows from the theorem applied to the ω-functors $\varnothing \to X$ and $\varnothing \to Y$. □

Remark 3.12. — This corollary was first proved directly by Hadzihasanovic (see [**8**, Theorem 1.35]).

4. A cylinder object for the folk model category structure

In paragraph 1.7, we introduced an ω-category J_1. The goal of this section is to prove that if X is a cofibrant ω-category, then $\mathrm{J}_1 \otimes X$ is a cylinder object for X in the folk model category structure.

We will start by showing that the tensor product of a reversible cell by any other n-cell is reversible, dealing first with the case $n = 1$. The proof will be a bit involved and we begin by the following technical lemmas:

Lemma 4.1. — *Let $u : x \to y$ be a reversible n-cell of an ω-category X. Fix $\bar{u}$ a reverse of u and $\varepsilon : u *_{n-1} \bar{u} \to 1_y$ a reversible cell. Then there exists a reversible n-cell $\eta : 1_x \to \bar{u} *_{n-1} u$ for which there exists a reversible $(n+1)$-cell $(\varepsilon *_{n-1} u) *_n (u *_{n-1} \eta) \to 1_u$.*

Proof. — Considering the n-cell $u : x \to y$, when $n > 1$, as a 1-cell of the ω-category of cells of X from sx to ty, we reduce to the case $n = 1$.

In this case, since u is reversible, there exists a reversible 2-cell $\eta' : 1_x \to \bar{u} *_0 u$. We set

$$\eta = (\bar{u} *_0 u *_0 \bar{\eta}') *_1 (\bar{u} *_0 \bar{\varepsilon} *_0 u) *_1 \eta',$$

where $\bar{\eta}'$ and $\bar{\varepsilon}$ denote reverses of η' and ε, respectively. The cell η is reversible, as a composite of reversible cells. Moreover, we have

$$\begin{aligned}
&(\varepsilon *_0 u) *_1 (u *_0 \eta) \\
&\quad = (\varepsilon *_0 u) *_1 (u *_0 \bar{u} *_0 u *_0 \bar{\eta}') *_1 (u *_0 \bar{u} *_0 \bar{\varepsilon} *_0 u) *_1 (u *_0 \eta') \\
&\quad = (u *_0 \bar{\eta}') *_1 (\varepsilon *_0 u *_0 \bar{u} *_0 u) *_1 (u *_0 \bar{u} *_0 \bar{\varepsilon} *_0 u) *_1 (u *_0 \eta') \\
&\quad = (u *_0 \bar{\eta}') *_1 ((\bar{\varepsilon} *_1 \varepsilon) *_0 u) *_1 (u *_0 \eta') \\
&\quad \to (u *_0 \bar{\eta}') *_1 (u *_0 \eta) = u *_0 (\bar{\eta}' *_1 \eta) \to 1_u,
\end{aligned}$$

where the two arrows are reversible 2-cells coming from the fact that $\bar{\varepsilon}$ and $\bar{\eta}'$ are reverses of ε and η', respectively, which concludes the proof of the lemma. ☐

Lemma 4.2. — *Let X be an ω-category and let c be an n-cell of $\Gamma(X)$. If c is reversible in $\Gamma(X)$, then the principal cell of c is reversible in X.*

Proof. — In this proof, we will use freely the explicit formulas for the structure of ω-category of $\Gamma(X)$, as given for instance in [**2**, Proposition B.1.15]. We prove the lemma by coinduction (see paragraph 1.2). Let $c = (x, y, \alpha)$ be an n-cell of $\Gamma(X)$ (see paragraph 1.13). We have

$$\alpha_n : \alpha^+_{n-1} *_{n-1} \alpha^+_{n-2} *_{n-2} \cdots *_1 \alpha^+_0 *_0 x \to y *_0 \alpha^-_0 *_1 \cdots *_{n-1} \alpha^-_{n-1}.$$

Suppose that (x, y, α) is reversible in $\Gamma(X)$ and let $(\bar{x}, \bar{y}, \beta)$ be a reverse. Note that $\bar{x}$ and $\bar{y}$ are reverses of x and y, respectively. The relationship between the source and target of (x, y, α) and $(\bar{x}, \bar{y}, \beta)$ implies that

$$\beta^\varepsilon_k = \alpha^\varepsilon_k \qquad \text{if } 0 \leqslant k < n-1 \text{ and } \varepsilon = \pm,$$
$$\beta^-_{n-1} = \alpha^+_{n-1} \quad \text{and} \quad \beta^+_{n-1} = \alpha^+_{n-1},$$
$$\beta_n : \alpha^-_{n-1} *_{n-1} \alpha^+_{n-2} *_{n-2} \cdots *_1 \alpha^+_0 *_0 \bar{x} \to \bar{y} *_0 \alpha^-_0 *_1 \cdots *_{n-2} \alpha^-_{n-2} *_{n-1} \alpha^+_{n-1}.$$

By hypothesis, there exists a reversible cell

$$(\varepsilon_x, \varepsilon_y, \Lambda) : (x, y, \alpha) *_{n-1} (\bar{x}, \bar{y}, \beta) \to 1_{t(x,y,\alpha)}.$$

In particular, the cells

$$\varepsilon_x : x *_{n-1} \bar{x} \to 1_{tx} \quad \text{and} \quad \varepsilon_y : y *_{n-1} \bar{y} \to 1_{ty}$$

are reversible. The cell $(x, y, \alpha) *_{n-1} (\bar{x}, \bar{y}, \beta)$ is of the form $(x *_{n-1} \bar{x}, y *_{n-1} \bar{y}, \gamma)$, with

$$\begin{gathered}\gamma_k^\varepsilon = \alpha_k^\varepsilon \qquad \text{if } 0 \leqslant k < n-1 \text{ and } \varepsilon = \pm, \\ \gamma_{n-1}^- = \beta_{n-1}^- = \alpha_{n-1}^+ \quad \text{and} \quad \gamma_{n-1}^+ = \alpha_{n-1}^+, \\ \gamma_n = (y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^- *_{n-1} \beta_n) *_n (\alpha_n *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \bar{x})\end{gathered}$$

and we have

$$\Lambda_{n+1} : \alpha_{n-1}^+ *_{n-1} \cdots *_1 \alpha_0^+ *_0 \varepsilon_x \to \varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^- *_{n-1} \alpha_{n-1}^+ *_n \gamma_n.$$

Applying Lemma 4.1 to the cell $(\varepsilon_x, \varepsilon_y, \Lambda)$ gives a reversible cell

$$(\eta_x, \eta_y, \Gamma) : 1_{s(x,y,\alpha)} \to (\bar{x}, \bar{y}, \beta) *_{n-1} (x, y, \alpha)$$

and, using the projections π^- and π^+, reversible cells

$$(\varepsilon_x *_{n-1} x) *_n (x *_{n-1} \eta_x) \to 1_x \quad \text{and} \quad (\varepsilon_y *_{n-1} y) *_n (y *_{n-1} \eta_y) \to 1_y.$$

The cell $(\bar{x}, \bar{y}, \beta) *_{n-1} (x, y, \alpha)$ is of the form $(\bar{x} *_{n-1} x, \bar{y} *_{n-1} y, \delta)$ with

$$\begin{gathered}\delta_k^\varepsilon = \alpha_k^\varepsilon \qquad \text{if } 0 \leqslant k < n-1 \text{ and } \varepsilon = \pm, \\ \delta_{n-1}^- = \alpha_{n-1}^- \quad \text{and} \quad \gamma_{n-1}^+ = \beta_{n-1}^+ = \alpha_{n-1}^-, \\ \delta_n = (\bar{y} *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^- *_{n-1} \alpha_n) *_n (\beta_n *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\end{gathered}$$

and we have

$$\Gamma_{n+1} : \delta_n *_n \alpha_{n-1}^- *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x \to \eta_y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-.$$

We now define our candidate n-cell ρ to be a reverse of α_n:

$$\begin{aligned}\rho = \big(&(\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_{n-1}^+ *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\ &*_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \beta_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\ &*_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big).\end{aligned}$$

We first produce a reversible cell between $\rho *_n \alpha_n$ and $1_{s(\alpha_n)}$. We have

$$\begin{aligned}&\rho *_n \alpha_n \\ &\quad = \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_{n-1}^+ *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\ &\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \beta_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\ &\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big) \\ &\qquad *_n \alpha_n \\ &\quad = \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_{n-1}^+ *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\ &\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \beta_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\ &\qquad *_n \big(\alpha_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 (\bar{x} *_{n-1} x))\big) \\ &\qquad *_n \big((\alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big)\end{aligned}$$

$$
\begin{aligned}
&= \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_{n-1}^+ *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \beta_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\alpha_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \bar{x}) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 (x *_{n-1} \eta_x)\big) \\
&= \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_{n-1}^+ *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\big[(y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^- *_{n-1} \beta_n) \\
&\qquad *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \bar{x})\big] *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 (x *_{n-1} \eta_x)\big) \\
&= \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_{n-1}^+ *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\gamma_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 (x *_{n-1} \eta_x)\big) \\
&= \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^- *_{n-1} \alpha_{n-1}^+ *_n \gamma_n) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 (x *_{n-1} \eta_x)\big).
\end{aligned}
$$

By coinduction, the cell

$$\Lambda_{n+1} : \alpha_{n-1}^+ *_{n-1} \cdots *_1 \alpha_0^+ *_0 \varepsilon_x \to \varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^- *_{n-1} \alpha_{n-1}^+ *_n \gamma_n$$

is reversible and we thus get a reversible cell between $\rho *_n \alpha_n$ and the cell

$$
\begin{aligned}
&\big((\alpha_{n-1}^+ *_{n-1} \cdots *_1 \alpha_0^+ *_0 \varepsilon_x) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\quad *_n \big(\alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 (x *_{n-1} \eta_x)\big) \\
&\quad = \alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \big((\varepsilon_x *_{n-1} x) *_n (x *_{n-1} \eta_x)\big)
\end{aligned}
$$

and hence a reversible cell between $\rho *_n \alpha_n$ and the identity on

$$\alpha_{n-1}^+ *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x = s(\alpha_n).$$

We now produce a reversible cell between $\alpha_n *_n \rho$ and $1_{t(\alpha_n)}$. We have

$$
\begin{aligned}
&\alpha_n *_n \rho \\
&\quad = \alpha_n \\
&\qquad *_n \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_{n-1}^+ *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \beta_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big) \\
&\quad = \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} (y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-)\big) \\
&\qquad *_n \big(((y *_{n-1} \bar{y}) *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \alpha_n\big) \\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} \beta_n *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big) \\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} (y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-)\big)\\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-2}^-) *_{n-1} \big[(\bar{y} *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^- *_{n-1} \alpha_n)\\
&\qquad\qquad *_n (\beta_n *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 x)\big]\big)\\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big)\\
&= \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} (y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-)\big)\\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-2}^-) *_{n-1} \delta_n\big)\\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-) *_{n-1} (\alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big)\\
&= \big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} (y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-)\big)\\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-2}^-)\\
&\qquad\qquad *_{n-1} (\delta_n *_n \alpha_{n-1}^- *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x)\big).
\end{aligned}
$$

By coinduction, the cell

$$\Gamma_{n+1} : \delta_n *_n \alpha_{n-1}^- *_{n-1} \alpha_{n-2}^+ *_{n-2} \cdots *_1 \alpha_0^+ *_0 \eta_x \to \eta_y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-$$

is reversible and we thus get a reversible cell from $\alpha_n * \rho$ to

$$
\begin{aligned}
&\big((\varepsilon_y *_0 \alpha_0^- *_1 \cdots *_{n-2} \alpha_{n-2}^-) *_{n-1} (y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-)\big)\\
&\qquad *_n \big((y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-2}^-) *_{n-1} (\eta_y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-)\big)\\
&\qquad = ((\varepsilon_y *_{n-1} y) *_n (y *_{n-1} \eta_y)) *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-
\end{aligned}
$$

and hence a reversible cell between $\alpha_n *_n \rho$ and the identity on

$$y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^- = t(\alpha_n),$$

hence the result. □

4.3. — We will denote by R_n the free-standing reversible n-cell and by $r_n : \mathrm{D}_n \to \mathrm{R}_n$ the canonical ω-functor. This ω-category R_n is freely generated by

– two n-cells
$$r : x \to y, \quad \bar{r} : y \to x,$$
– four $(n+1)$-cells
$$\alpha : \bar{r} *_{n-1} r \to 1_x,\ \bar{\alpha} : 1_x \to \bar{r} *_{n-1} r,\ \beta : r *_{n-1} \bar{r} \to y,\ \bar{\beta} : y \to r *_{n-1} \bar{r},$$
– eight $(n+2)$-cells comparing $\bar{\alpha} *_n \alpha$, $\alpha *_n \bar{\alpha}$, $\bar{\beta} *_n \beta$ and $\beta *_n \bar{\beta}$ to identities,
– etc. (see the remark below for a formal description).

The ω-functor $r_n : \mathrm{D}_n \to \mathrm{R}_n$ sends the principal cell of D_n to r, *the principal cell of* R_n. Note that this ω-functor is a folk cofibration. By definition, an n-cell x of an ω-category X is reversible if and only if the ω-functor $\langle x \rangle : \mathrm{D}_n \to X$ factors through r_n. Note that such a factorization corresponds to a choice of witnesses that r is reversible and is hence not unique.

Remark 4.4. — More formally, the ω-category R_n is generated by two n-cells

$$r : x \to y, \quad \bar{r} : y \to x,$$

and, for every $i > n$, two sets of 2^{i-n} i-cells

$$r_{l_1,\dots,l_{i-n}} \quad \text{and} \quad \bar{r}_{l_1,\dots,l_{i-n}},$$

where $l_j = \pm 1$ for $1 \leqslant j \leqslant i-n$, whose sources and targets are given by

$$\begin{aligned}
r_{l_1,\dots,l_{i-n-1},-} &: \quad \bar{r}_{l_1,\dots,l_{i-n-1}} *_{i-1} r_{l_1,\dots,l_{i-n-1}} \to 1_{s(r_{l_1,\dots,l_{i-n-1}})}\\
\bar{r}_{l_1,\dots,l_{i-n-1},-} &: \quad 1_{s(r_{l_1,\dots,l_{i-n-1}})} \to \bar{r}_{l_1,\dots,l_{i-n-1}} *_{i-1} r_{l_1,\dots,l_{i-n-1}}\\
r_{l_1,\dots,l_{i-n-1},+} &: \quad r_{l_1,\dots,l_{i-n-1}} *_{i-1} \bar{r}_{l_1,\dots,l_{i-n-1}} \to 1_{t(r_{l_1,\dots,l_{i-n-1}})}\\
\bar{r}_{l_1,\dots,l_{i-n-1},+} &: \quad 1_{t(r_{l_1,\dots,l_{i-n-1}})} \to r_{l_1,\dots,l_{i-n-1}} *_{i-1} \bar{r}_{l_1,\dots,l_{i-n-1}} \quad .
\end{aligned}$$

(With this notation, the $(n+1)$-cells α and β of the previous paragraph are $\alpha = r_-$ and $\beta = r_+$.)

Proposition 4.5. — *If x is a 1-cell of an ω-category X and y is a reversible n-cell of an ω-category Y, then $x \otimes y$ is reversible in $X \otimes Y$.*

Proof. — Since y is reversible, $\langle y \rangle : \mathrm{D}_n \to Y$ factors through $r_n : \mathrm{D}_n \to \mathrm{R}_n$. Tensoring by $\langle x \rangle$, we therefore get an ω-functor $\mathrm{D}_1 \otimes \mathrm{R}_n \to X \otimes Y$, which corresponds by adjunction to an ω-functor $\mathrm{R}_n \to \Gamma(X \otimes Y)$ and hence to a reversible cell of $\Gamma(X \otimes Y)$. By Lemma 4.2, the principal cell of this cylinder is reversible in $X \otimes Y$. But this principal cell corresponds to the composite

$$\mathrm{D}_{n+1} \xrightarrow{\langle p \rangle} \mathrm{D}_1 \otimes \mathrm{D}_n \xrightarrow{\langle x \rangle \otimes \langle y \rangle} X \otimes Y,$$

where p denotes the principal cell of $\mathrm{D}_1 \otimes \mathrm{D}_n$, which is $\langle x \otimes y \rangle$. This shows that $x \otimes y$ is reversible. □

Corollary 4.6. — *If x is a reversible n-cell of an ω-category X and y is a 1-cell of an ω-category Y, then $x \otimes y$ is reversible in $X \otimes Y$.*

Proof. — We will use the duality $Z \mapsto Z^{\mathrm{op}}$ introduced in paragraph 1.20, denoting by z^{op} the cell of Z^{op} corresponding to a cell z of Z. By this same paragraph, if x is a reversible cell of X and y is a 1-cell of Y, then $x \otimes y$ is reversible in $X \otimes Y$ if and only if $(x \otimes y)^{\mathrm{op}}$ is reversible in $(X \otimes Y)^{\mathrm{op}}$ if and only if $y^{\mathrm{op}} \otimes x^{\mathrm{op}}$ is reversible in $Y^{\mathrm{op}} \otimes X^{\mathrm{op}}$. As x^{op} is reversible in X^{op}, the result thus follows from the previous proposition. □

We will now show that the tensor product of a reversible cell by any cell is reversible. We will need a specific ω-functor from $\mathrm{D}_{n-1} \otimes \mathrm{D}_1$ to D_n that we now introduce.

4.7. — Let $n \geqslant 1$. The ω-category obtained from $\mathrm{D}_{n-1} \otimes \mathrm{D}_1$ by collapsing, independently, the sub-ω-categories $\mathrm{D}_{n-1} \otimes \{0\}$ and $\mathrm{D}_{n-1} \otimes \{1\}$ is canonically isomorphic to D_n. (This follows for instance from [**2**, Corollary B.6.6] using the duality $X \mapsto X^{\mathrm{co}}$.) In particular, there is a canonical ω-functor

$$\mathrm{D}_{n-1} \otimes \mathrm{D}_1 \to \mathrm{D}_n$$

sending $\mathrm{D}_{n-1} \otimes \{0\}$ to 0, $\mathrm{D}_{n-1} \otimes \{1\}$ to 1, and the principal cell of $\mathrm{D}_{n-1} \otimes \mathrm{D}_1$ to the principal cell of D_n.

By iterating this construction, we get an ω-functor

$$\mathrm{D}_1^{\otimes n} \to \mathrm{D}_n$$

sending *the principal cell of* $\mathrm{D}_1^{\otimes n}$, that is, the tensor product of the principal cells of the n copies of D_1, to the principal cell of D_n.

Lemma 4.8. — *If r is the principal cell of R_k and d is the principal cell of D_n, then $r \otimes d$ is reversible in $\mathrm{R}_k \otimes \mathrm{D}_n$.*

Proof. — By the Corollary 4.6, for every $m \geqslant 0$, the tensor product of the principal cells of R_m and D_1 is reversible, showing that there exists an ω-functor $p' : \mathrm{R}_{m+1} \to \mathrm{R}_m \otimes \mathrm{D}_1$ making commutative the square

$$\begin{tikzcd}
\mathrm{D}_{m+1} \arrow[r, "r_{m+1}"] \arrow[d, "p"'] & \mathrm{R}_{m+1} \arrow[d, "p'"] \\
\mathrm{D}_m \otimes \mathrm{D}_1 \arrow[r, "r_m \otimes \mathrm{D}_1"'] & \mathrm{R}_m \otimes \mathrm{D}_1
\end{tikzcd} ,$$

where p corresponds to the principal cell of $\mathrm{D}_m \otimes \mathrm{D}_1$. Denote this square by S_m. By composing the n squares $S_{n+k-1-m} \otimes \mathrm{D}_1^{\otimes m}$ for $0 \leqslant m \leqslant n-1$

$$\begin{tikzcd}
\mathrm{D}_{n+k} \arrow[r] \arrow[d] & \mathrm{R}_{n+k} \arrow[d] \\
\mathrm{D}_{n+k-1} \otimes \mathrm{D}_1 \arrow[r] \arrow[d] & \mathrm{R}_{n+k-1} \otimes \mathrm{D}_1 \arrow[d] \\
(\mathrm{D}_{n+k-2} \otimes \mathrm{D}_1) \otimes \mathrm{D}_1 \arrow[r] \arrow[d, dotted] & (\mathrm{R}_{n+k-2} \otimes \mathrm{D}_1) \otimes \mathrm{D}_1 \arrow[d, dotted] \\
(\mathrm{D}_k \otimes \mathrm{D}_1) \otimes \mathrm{D}_1^{\otimes n-1} \arrow[r] & (\mathrm{R}_k \otimes \mathrm{D}_1) \otimes \mathrm{D}_1^{\otimes n-1}
\end{tikzcd} ,$$

we get a commutative square

$$\begin{tikzcd}
\mathrm{D}_{n+k} \arrow[r, "r_{n+k}"] \arrow[d] & \mathrm{R}_{n+k} \arrow[d] \\
\mathrm{D}_k \otimes \mathrm{D}_1^{\otimes n} \arrow[r, "r_k \otimes \mathrm{D}_1^{\otimes n}"'] & \mathrm{R}_k \otimes \mathrm{D}_1^{\otimes n}
\end{tikzcd} .$$

By composing this square with the ω-functor $\mathrm{D}_1^{\otimes n} \to \mathrm{D}_n$ of the previous paragraph, we get a commutative square

$$\begin{tikzcd}
\mathrm{D}_{n+k} \arrow[r, "r_{n+k}"] \arrow[d] & \mathrm{R}_{n+k} \arrow[d] \\
\mathrm{D}_k \otimes \mathrm{D}_n \arrow[r, "r_k \otimes \mathrm{D}_n"'] & \mathrm{R}_k \otimes \mathrm{D}_n
\end{tikzcd}$$

showing that $\langle r \otimes d \rangle$ factors through r_{n+k} and hence that $r \otimes d$ is reversible. $\square$

Proposition 4.9. — *Let x be an m-cell of an ω-category X and let y be an n-cell of an ω-category Y. If either x or y is reversible, then $x \otimes y$ is reversible in $X \otimes Y$.*

Proof. — We start with the case where x is reversible. The cell $x \otimes y$ then corresponds to the composite

$$\langle x \otimes y \rangle : \mathrm{D}_{m+n} \xrightarrow{\langle r \otimes d \rangle} \mathrm{R}_m \otimes \mathrm{D}_n \xrightarrow{\langle x \rangle \otimes \langle y \rangle} X \otimes Y,$$

where r and d denotes the principal cells of R_m and D_n respectively. But by the previous lemma, $r \otimes d$ is reversible, and thus so is

$$x \otimes y = (\langle x \rangle \otimes \langle y \rangle)(r \otimes d).$$

Suppose now that y is reversible. Then y^{op} is reversible, and so is $y^{\mathrm{op}} \otimes x^{\mathrm{op}}$ in $Y^{\mathrm{op}} \otimes X^{\mathrm{op}}$ by the previous case. This proves that $x \otimes y = (y^{\mathrm{op}} \otimes x^{\mathrm{op}})^{\mathrm{op}}$ is reversible. $\square$

We will use the previous proposition to study the notion of a J_1-transformation that we now introduce.

4.10. — Let $f, g : X \to Y$ be two ω-functors. A J_1*-transformation* from f to g is an ω-functor

$$h : \mathrm{J}_1 \otimes X \to Y$$

making commutative the diagram

$$\begin{tikzcd}
X \arrow[dr, "\langle 0 \rangle \otimes X"'] \arrow[drr, bend left, "f"] & & \\
& \mathrm{J}_1 \otimes X \arrow[r, "h"] & Y \\
X \arrow[ur, "\langle 1 \rangle \otimes X"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd},$$

where we denoted by 0 and 1 the image by $l_1 : \mathrm{D}_1 \to \mathrm{J}_1$ (see paragraph 1.7) of the objects 0 and 1 of D_1.

Note that if $h : \mathrm{J}_1 \otimes X \to Y$ is a J_1-transformation then the composite

$$\mathrm{D}_1 \otimes X \xrightarrow{l_1 \otimes X} \mathrm{J}_1 \otimes X \xrightarrow{h} Y$$

defines an oplax transformation from f to g.

Proposition 4.11. — *The oplax transformation associated to a* J_1*-transformation is reversible.*

Proof. — Let $h : \mathrm{J}_1 \otimes X \to Y$ be a J_1-transformation and let x be an n-cell of X. We have to show that $h((01) \otimes x)$, where (01) denotes the principal cell of J_1 (see paragraph 1.7) is reversible. It suffices to show that $(01) \otimes x$ is reversible and hence, by Proposition 4.9, that (01) is reversible in J_1. As the ω-functor $r_1 : \mathrm{D}_1 \to \mathrm{R}_1$ is a cofibration and the ω-functor $q_1 : \mathrm{J}_1 \to \mathrm{D}_0$ is a trivial fibration, the commutative square

$$\begin{tikzcd} \mathrm{D}_1 \arrow[r, "l_1"] \arrow[d, "r_1"'] & \mathrm{J}_1 \arrow[d, "q_1"] \\ \mathrm{R}_1 \arrow[r] \arrow[ur, dotted] & \mathrm{D}_0 \end{tikzcd}$$

admits a lift, showing that (01) is indeed reversible in J_1. □

4.12. — We say that an ω-functor $i : X \to Y$ is an *oplax transformation retract* (resp. a *reversible transformation retract*, resp. a J_1*-transformation retract*) if it admits a retraction r, that is, an ω-functor $r : Y \to X$ such that $ri = 1_X$, and an oplax transformation (resp. a reversible transformation, resp. a J_1-transformation) $\alpha : ir \Rightarrow 1_Y$.

It follows from the fact that reversible transformations are right homotopies for the folk model category structure (see Remark 1.17) that a reversible transformation retract is a folk weak equivalence.

We say that a transformation retract $i : X \to Y$ (oplax, reversible or J_1-) is *strong* if r and α as above can be chosen so that $\alpha * i = 1_i$, in the sense that, if α is given by an ω-functor

$$h : \mathrm{D}_1 \otimes Y \to Y \quad \text{or} \quad h' : \mathrm{J}_1 \otimes Y \to Y,$$

then the diagrams

$$\begin{tikzcd} \mathrm{D}_1 \otimes X \arrow[r, "p"] \arrow[d, "\mathrm{D}_1 \otimes i"'] & X \arrow[d, "i"] \\ \mathrm{D}_1 \otimes Y \arrow[r, "h"'] & Y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \mathrm{J}_1 \otimes X \arrow[r, "p'"] \arrow[d, "\mathrm{J}_1 \otimes i"'] & X \arrow[d, "i"] \\ \mathrm{J}_1 \otimes Y \arrow[r, "h'"'] & Y \end{tikzcd} \quad ,$$

where p and p' are the "projection" ω-functors induced by $\mathrm{D}_1 \to \mathrm{D}_0$ and $\mathrm{J}_1 \to \mathrm{D}_0$, commute.

By [**15**, Corollary 4.30], every folk trivial cofibration is a strong reversible transformation retract. (Note that strong reversible transformation retracts are called "immersions" in [**15**].)

Proposition 4.13. — *A* J_1*-transformation retract is a reversible transformation retract and in particular a folk weak equivalence.*

Proof. — This follows immediately from Proposition 4.11. $\square$

Proposition 4.14. — *For every ω-category X, the ω-functors $X \to \mathrm{J}_1 \otimes X$, obtained by tensoring $\langle 0\rangle, \langle 1\rangle : \mathrm{D}_0 \to \mathrm{J}_1$ by X, are J_1-transformation retracts and hence folk weak equivalences.*

Proof. — The class of J_1-transformation retracts is clearly stable under tensor product by an object on the right. Therefore it suffices to prove that $\langle 0\rangle, \langle 1\rangle : \mathrm{D}_0 \to \mathrm{J}_1$ are J_1-transformation retracts. Let $\varepsilon = 0, 1$. The ω-functor $r : \mathrm{J}_1 \to \mathrm{D}_0$ is clearly a retraction of $\langle\varepsilon\rangle$. Moreover, a J_1-transformation $\langle\varepsilon\rangle r \Rightarrow 1_{\mathrm{J}_1}$ is precisely a lift to the lifting problem

$$\begin{tikzcd} \partial\mathrm{D}_1 \otimes \mathrm{J}_1 \arrow[r, "\langle\varepsilon\rangle r \amalg \mathrm{J}_1"] \arrow[d, "k_1 \otimes \mathrm{J}_1"'] & \mathrm{J}_1 \arrow[d] \\ \mathrm{J}_1 \otimes \mathrm{J}_1 \arrow[r] \arrow[ur, dotted] & \mathrm{D}_0 \end{tikzcd} ,$$

where we identified $\partial\mathrm{D}_1 \otimes \mathrm{J}_1$ with $\mathrm{J}_1 \amalg \mathrm{J}_1$. As k_1 is a cofibration and J_1 is a cofibrant object, it follows from Theorem 3.9 that the left vertical arrow is a cofibration. As the right vertical arrow is a trivial fibration by definition of J_1, the desired lift exists, thereby proving the result. $\square$

Theorem 4.15. — *Let X be an ω-category.*

(a) *The ω-functor $\mathrm{J}_1 \otimes X \to X$, obtained by tensoring $\mathrm{J}_1 \to \mathrm{D}_0$ by X, is a folk weak equivalence.*

(b) *If moreover, X is cofibrant, then the factorization*

$$X \amalg X \to \mathrm{J}_1 \otimes X \to X$$

of the codiagonal, obtained by tensoring

$$\partial\mathrm{D}_1 \xrightarrow{k_1} \mathrm{J}_1 \to \mathrm{D}_0$$

by X, is a cylinder object for the folk model category structure, in the sense that the first arrow is a cofibration and the second one is a weak equivalence.

Proof. — The ω-functor $\mathrm{J}_1 \otimes X \to X$ is a retract of any of the ω-functors $X \to \mathrm{J}_1 \otimes X$ considered in the previous proposition and is hence a weak equivalence by this same proposition. This proves the first point. As for the second one, it follows from Theorem 3.9 since k_1 is a cofibration and X is cofibrant. $\square$

Remark 4.16. — The ω-functor $X \amalg X \to \mathrm{J}_1 \otimes X$ is not a folk cofibration in general. To see this, recall first that if $p : y_0 \to y_1$ is a 1-cell in an ω-category Y and $f : x_0 \to x_1$, $g : x_1 \to x_2$ are two 1-cells in an ω-category X, then the following relation holds in $Y \otimes X$:

$$((y_1 \otimes g) *_0 (p \otimes f)) *_1 ((p \otimes g) *_0 (y_0 \otimes f)) = p \otimes (g *_0 f).$$

Diagrammatically, this means that the following 2-cells are equal:

$$
\begin{tikzcd}
y_0 \otimes x_0 \arrow[r, "y_0 \otimes f"] \arrow[d, "p \otimes x_0"'] \arrow[dr, phantom, "\swarrow p\otimes f"] & y_0 \otimes x_1 \arrow[r, "y_0 \otimes g"] \arrow[d, "p \otimes x_1"'] \arrow[dr, phantom, "\swarrow p\otimes g"] & y_0 \otimes x_2 \arrow[d, "p \otimes x_2"] \\
y_1 \otimes x_0 \arrow[r, "y_1 \otimes f"'] & y_1 \otimes x_1 \arrow[r, "y_1 \otimes g"'] & y_1 \otimes x_2
\end{tikzcd}
\qquad
\begin{tikzcd}
y_0 \otimes x_0 \arrow[r, "y_0 \otimes (g *_0 f)"] \arrow[d, "p \otimes x_0"'] \arrow[dr, phantom, "\swarrow p\otimes (g *_0 f)"] & y_0 \otimes x_2 \arrow[d, "p \otimes x_2"] \\
y_1 \otimes x_0 \arrow[r, "y_1 \otimes (g *_0 f)"'] & y_1 \otimes x_2
\end{tikzcd}
.
$$

Now take X to be the category generated by one object x and one arrow $f : x \to x$, subject to the relation $f *_0 f = 1_x$. In other words, X is the cyclic group with 2 elements, seen as a one-object category. Let $p : y_0 \to y_1$ be the principal cell of J_1. Then the previous formula (taking $g = f$) shows that the following equality holds in $\mathrm{J}_1 \otimes X$:

$$((y_1 \otimes f) *_0 (p \otimes f)) *_1 ((p \otimes f) *_0 (y_0 \otimes f)) = 1_{p \otimes x}.$$

This relation implies that any ω-functor from $\mathrm{J}_1 \otimes X$ to a cofibrant ω-category must send $p \otimes f$ to an identity. Consider now $u : \mathrm{J}_1 \otimes X \to Z$ the pushout of the obvious ω-functor $X \amalg X \to \mathrm{D}_0 \amalg \mathrm{D}_0 = \partial\mathrm{D}_1$ along the ω-functor $X \amalg X \to \mathrm{J}_1 \otimes X$. One can check that the cell $u(p \otimes f)$ is non-trivial in Z. Factoring the map $\partial\mathrm{D}_1 \to Z$ into a cofibration followed by a trivial fibration $t : Z' \to Z$, we get a commutative square

$$
\begin{tikzcd}
X \amalg X \arrow[r] \arrow[d] & Z' \arrow[d, "t"] \\
\mathrm{J}_1 \otimes X \arrow[r, "u"'] & Z
\end{tikzcd}
,
$$

with Z' a cofibrant ω-category. But this square cannot admit a lift for such a lift would map $p \otimes f$ to an identity in Z', contradicting the fact that $u(p \otimes f)$ is non-trivial. Therefore the map $X \amalg X \to \mathrm{J}_1 \otimes X$ is not a cofibration.

Corollary 4.17. — *Let $f, g : X \to Y$ be two ω-functors, where X is a cofibrant ω-category. If there exists a reversible transformation from f to g, then there exists a J_1-transformation from f to g.*

Proof. — As all the ω-categories are fibrant for the folk model category structure and X is cofibrant by hypothesis, the relations of left homotopy and right homotopy on $\mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X, Y)$ coincide. We get the result using the path object $\Gamma_{\mathrm{rev}}(Y)$ (see Theorem 1.16) and the cylinder object $\mathrm{J}_1 \otimes X$ given by the previous theorem. □

Corollary 4.18. — *Any reversible transformation retract whose target is cofibrant is a J_1-transformation retract.*

Proof. — This follows immediately from the previous corollary. □

Remark 4.19. — The following diagram sums up the relationship between the different classes of ω-functors considered in this section in between folk trivial cofibrations and folk weak equivalences:

$$\begin{tikzcd}
& \text{folk trivial cofibration} \arrow[dl, Rightarrow, "\text{Prop 5.7}"'] \arrow[d, Rightarrow, "{[\mathbf{15}, \text{Cor 4}]}"] & \\
\mathrm{J}_1\text{-transformation retract} \arrow[r, Rightarrow, shift left, "\text{Prop 4.13}"] & \text{reversible transformation retract} \arrow[l, Rightarrow, dotted, shift left, "\text{Cor 4.18}"] \arrow[r, Rightarrow, "{[\mathbf{15}, \text{Lem 16}]}"'] & \text{folk weak equivalence}
\end{tikzcd}.$$

(The dotted arrow means that the implication holds under the additional assumption that the target is cofibrant.)

5. The folk model category structure is monoidal for the tensor product

In this section, we will end the proof of the compatibility of the Gray tensor product with the folk model category structure and give some supplements on the resulting monoidal model category.

We start by a general lemma abstracting our strategy to prove this compatibility:

Lemma 5.1. — *Let $\mathcal{M}$ be a cofibrantly generated model category endowed with a (not necessarily symmetric nor closed) monoidal category structure satisfying the following hypotheses:*

H1) the unit of the tensor product $\otimes$ is cofibrant,

H2) the sources of the generating cofibrations are cofibrant,

H3) if i is a cofibration (resp. *a trivial cofibration*)*, then so are $i \otimes \varnothing$ and $\varnothing \otimes i$, where $\varnothing$ denotes the initial object of $\mathcal{M}$,*

H4) for every generating cofibrations $i : A \to B$ and $j : C \to D$, the pushout-product

$$i \otimes' j : B \otimes C \amalg_{A \otimes C} A \otimes D \to B \otimes D$$

is a cofibration,

H5) for every generating trivial cofibration i and every cofibrant object A, the morphisms $i \otimes A$ and $A \otimes i$ are weak equivalences.

Then $\mathcal{M}$ is a monoidal model category.

Proof. — First note that by Remark A.2, the hypothesis H1) and H3) imply that the unit axiom (see paragraph A.1) is satisfied.

Moreover, by the classical Lemma A.3 (see also Remark A.4), the pushout-product axiom (see again paragraph A.1) can be checked on generators. The hypothesis H4) thus implies that the pushout-product of two cofibrations is a cofibration, and it suffices to show that if, either $i : A \to B$ is a generating trivial cofibration and $j : C \to D$ is a generating cofibration, or i is a generating cofibration and j is a generating trivial cofibration, then $i \otimes' j$ is a weak equivalence. Let us prove the first case, the proof of the second one being dual.

Consider the commutative diagram

$$\begin{tikzcd}
A \otimes C \arrow[r, "A \otimes j"] \arrow[d, "i \otimes C"'] & A \otimes D \arrow[d, "\varepsilon_2"] \arrow[rdd, bend left, "i \otimes D"] & \\
B \otimes C \arrow[r, "\varepsilon_1"] \arrow[rrd, bend right, "B \otimes j"'] & B \otimes C \amalg_{A \otimes C} A \otimes D \arrow[rd, "i \otimes' j"] & \\
& & B \otimes D
\end{tikzcd} ,$$

where ε_1 and ε_2 are the canonical morphisms. The pushout-product axiom for cofibrations and hypothesis H3) imply that the tensor product of a cofibration and a cofibrant object is a cofibration (see Remark A.2). It thus follows from H2) and H5) that $i \otimes C$ is a trivial cofibration. The morphism ε_2 being a pushout of $i \otimes C$, it is a trivial cofibration as well. For the same reasons as above, the morphism $i \otimes D$ is a weak equivalence and the two-out-of-three property implies that $i \otimes' j$ is also a weak equivalence, thereby proving the lemma. ☐

Remark 5.2. — Note that under the hypothesis H1), H2) and H3) of the lemma, the fact that $\mathcal{M}$ is a monoidal model category is actually equivalent to the hypothesis H4) and H5).

Remark 5.3. — The hypothesis H1) and H2) are fulfilled by the folk model category structure. Moreover, the hypothesis H3) is fulfilled in any biclosed monoidal structure (we will also apply this lemma to the join of ω-categories which is not biclosed). To prove that the folk model category structure is monoidal for the Gray tensor product, it thus suffices to prove H4) and H5). The hypothesis H4) is Theorem 3.9 and will now prove H5).

Let us see that H5) is a direct consequence of results from the previous section.

Proposition 5.4. — *Let $i : X \to Y$ be a folk trivial cofibration between cofibrant objects. Then, for any ω-category Z, the ω-functor $i \otimes Z : X \otimes Z \to Y \otimes Z$ is a* J_1*-transformation retract and in particular a folk weak equivalence.*

Proof. — As noted in paragraph 4.12, it is proved in [**15**] that such an ω-functor i is a reversible transformation retract. As Y is cofibrant, Corollary 4.18 implies that i is a J_1-transformation retract. But it is immediate that J_1-transformation retracts are stable by tensoring by an object on the right, hence the result by Proposition 4.13. ☐

Corollary 5.5. — *Let $i : X \to Y$ be a folk trivial cofibration between cofibrant objects. Then, for any ω-category Z, the ω-functor $Z \otimes i : Z \otimes X \to Z \otimes Y$ is a folk weak equivalence.*

Proof. — We will use the duality $T \mapsto T^{\mathrm{op}}$ introduced in paragraph 1.20. By this same paragraph, this duality preserves cofibrations, trivial cofibrations and weak equivalences, and we have a natural isomorphism $(A \otimes B)^{\mathrm{op}} \simeq B^{\mathrm{op}} \otimes A^{\mathrm{op}}$. This implies that the ω-functor $i^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a trivial cofibration between cofibrant objects and hence, by the previous proposition, that $i^{\mathrm{op}} \otimes Z^{\mathrm{op}} : X^{\mathrm{op}} \otimes Z^{\mathrm{op}} \to Y^{\mathrm{op}} \otimes Z^{\mathrm{op}}$ is a folk weak equivalence. This shows that $Z \otimes i$, that can be identified with $(i^{\mathrm{op}} \otimes Z^{\mathrm{op}})^{\mathrm{op}}$, is indeed a folk weak equivalence. □

Theorem 5.6. — *The folk model category structure on ω-$\mathcal{C}at$ is monoidal for the Gray tensor product.*

Proof. — This follows from Lemma 5.1, whose non-trivial hypothesis are fulfilled by Theorem 3.9, and the previous proposition and its corollary. □

The previous theorem implies that the tensor product of a folk trivial cofibration and a cofibrant object is a weak equivalence. We will now prove that this still holds if we remove the cofibrancy hypothesis.

Proposition 5.7. — *Any folk trivial cofibration is a strong* J_1*-transformation retract.*

Proof. — Let $i : X \to Y$ be a trivial cofibration. As every ω-category is fibrant, the lifting problem

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

admits a solution $r : Y \to X$ giving a retraction of i. Similarly, by Theorem 5.6, the lifting problem

$$\begin{tikzcd} \mathrm{J}_1 \otimes X \amalg_{\partial\mathrm{D}_1 \otimes X} \partial\mathrm{D}_1 \otimes Y \arrow[r, "{(ip,(ir,1_Y))}"] \arrow[d, "k_1 \otimes' i"'] & Y \\ \mathrm{J}_1 \otimes Y \arrow[ur, dotted] & \end{tikzcd},$$

where p denotes the "projection" ω-functor $p : \mathrm{J}_1 \otimes X \to X$, admits a solution $h : \mathrm{J}_1 \otimes Y \to Y$. Such an h is precisely a J_1-transformation as in the definition of a strong J_1-transformation retract. □

Proposition 5.8. — *Strong* J_1*-transformation retracts are stable under pushouts.*

Proof. — The analogous statement for strong reversible transformation retracts is [**15**, Lemma 17], whose proof applies *mutatis mutandis.* □

Proposition 5.9. —
(a) *Transfinite compositions of pushouts of tensor products of an object (on the left) and a folk trivial cofibration are folk weak equivalences.*
(b) *Transfinite compositions of pushouts of tensor products of a folk trivial cofibration and an object (on the right) are folk weak equivalences.*

Proof. — The second assertion can be deduced from the first one using the duality $X \mapsto X^{\mathrm{op}}$ as in the proof of Corollary 5.5. As for the first one, by Proposition 5.7, trivial cofibrations are strong J_1-transformation retracts. But J_1-transformation retracts are stable by tensoring by an object on the left, essentially by definition, and by pushouts by the previous proposition. As J_1-transformation retracts are weak equivalences (by Proposition 4.14), the result follows from the fact that folk weak equivalences are stable under transfinite compositions (see [**15**, Lemma 4.12]). ☐

Remark 5.10. — In particular, the tensor product of a folk trivial cofibration by an object (on the left or on the right) is a folk weak equivalence.

Remark 5.11. — We proved more precisely that the ω-functors of the first assertion of the proposition are transfinite compositions of J_1-transformation retracts.

6. The case of (m, n)-categories

In this section, we fix m and n such that $0 \leqslant n \leqslant m \leqslant \omega$.

6.1. — Recall that an (m, n)*-category* is an ω-category X such that
- X is an m-category, that is, every k-cell of X with $k > m$ is an identity,
- every k-cell x of X, for $k > n$, is *invertible*, meaning that there exists a k-cell y such that
$$y \ast_{k-1} x = 1_{sx} \quad \text{and} \quad x \ast_{k-1} y = 1_{tx}.$$

We will denote by (m, n)-$\mathcal{C}at$ the full subcategory of ω-$\mathcal{C}at$ consisting of (m, n)-categories. Note that (m, m)-categories are nothing but m-categories, and $(m, 0)$-categories are m-groupoids, whose category will be denoted by m-$\mathcal{G}pd$.

The category (m, n)-$\mathcal{C}at$ is a reflective subcategory of ω-$\mathcal{C}at$. In other words, the inclusion functor (m, n)-$\mathcal{C}at \hookrightarrow \omega$-$\mathcal{C}at$ admits a left adjoint $r : \omega$-$\mathcal{C}at \to (m, n)$-$\mathcal{C}at$.

The goal of this section is to prove, first, that the Gray tensor product of ω-categories induces, using the ω-functor $r : \omega$-$\mathcal{C}at \to (m, n)$-$\mathcal{C}at$, a monoidal category structure on (m, n)-$\mathcal{C}at$ and, second, that this monoidal category structure is compatible with the folk model category structure on (m, n)-$\mathcal{C}at$.

To prove the first point, we will use Day's reflection theorem:

***Proposition 6.2* (Day).** — *Let $\mathcal{C}$ be a biclosed monoidal category and let $\mathcal{D} \subset \mathcal{C}$ be a reflective subcategory of $\mathcal{C}$. Then the following conditions are equivalent:*

(a) *for every object X of $\mathcal{C}$ and every object Y of $\mathcal{D}$, the objects $\underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{C}}(X,Y)$ and $\underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{C}}(X,Y)$ (see paragraph A.6) are in $\mathcal{D}$,*

(b) *for every objects X and Y of $\mathcal{C}$, the canonical morphism*

$$r(X \otimes Y) \xrightarrow{\sim} r(r(X) \otimes r(Y)),$$

where $r : \mathcal{C} \to \mathcal{D}$ denotes the left adjoint to the inclusion functor, is an isomorphism.

Moreover, when these conditions are satisfied, the tensor product

$$X \otimes_{\mathcal{D}} Y = r(X \otimes Y)$$

defines a biclosed monoidal category structure on $\mathcal{D}$, whose unit and internal Hom *are those of $\mathcal{C}$.*

Proof. — The analogous statement for closed symmetric monoidal categories is a particular case of [**6**, Theorem 1.2]. The proof applies *mutatis mutandis* to the case of biclosed monoidal categories. □

We will prove that condition (a) is satisfied in our case of interest, that is, that if X is an ω-category and Y is an (m,n)-category, then both $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X,Y)$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}(X,Y)$ are (m,n)-categories. We start by some preliminaries on invertible cells.

6.3. — We will denote by I_n the free-standing invertible n-cell in ω-$\mathcal{C}at$. In other words, I_n is the n-category obtained from D_n by formally inverting the principal cell of D_n. We have a canonical ω-functor $\mathrm{D}_n \to \mathrm{I}_n$. The image of the principal cell of D_n by this ω-functor is the *principal cell of* I_n. By definition, an n-cell x of an ω-category X is invertible if and only the corresponding ω-functor $\langle x \rangle : \mathrm{D}_n \to X$ factor through I_n.

Proposition 6.4. — *Let x be an m-cell of an ω-category X and let y be an n-cell of an ω-category Y. If either x or y is invertible, then $x \otimes y$ is invertible in $X \otimes Y$.*

Proof. — The proof is similar to the proof of the analogous fact for reversible cells (Proposition 4.9). More precisely, one first proves the statement analogous to Lemma 4.2 using the same calculations as in its proof and one then proves the statements analogous to Proposition 4.5, Corollary 4.6, Lemma 4.8 and, finally, Proposition 4.9, by a straightforward adaptation consisting essentially in replacing the ω-category R_n by I_n. □

6.5. — Let Y be an ω-category. We will say that an n-cylinder $c = (x, y, \alpha)$ (see paragraph 1.13) is *invertible* if all the α^{ε}_k, for $0 \leqslant k \leqslant n$ and $\varepsilon = \pm$, are invertible cells of Y. For the same reasons as for reversible cylinders, the graded subset $\Gamma_{\mathrm{inv}}(Y)$ of $\Gamma(Y)$ consisting of invertible cylinders forms a sub-ω-category.

We will now prove that the ω-functor

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y) \to \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, Y) = \Gamma(Y),$$

induced by the canonical ω-functor $\mathrm{D}_1 \to \mathrm{I}_1$, gives an isomorphism of ω-categories between $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)$ and $\Gamma_{\mathrm{inv}}(Y) \subset \Gamma(Y)$. (This will be achieved in Proposition 6.8.)

6.6. — Let Y be an ω-category. By adjunction, n-cells of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)$ correspond to ω-functors $\mathrm{I}_1 \to \underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_n, Y)$, that is, to invertible 1-cells of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_n, Y)$. Note that, again by adjunction, 1-cells of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_n, Y)$ corresponds to n-cylinders $c = (x, y, \alpha)$ in Y (and that the source and target of such a c correspond to the cells x and y respectively). This means that the composition of 1-cells in $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_n, Y)$ defines a composition on n-cylinders in Y, that we will call the *vertical composition*, and that the n-cells of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)$ correspond to the n-cylinders in Y invertible for the vertical composition. In particular, this shows that the ω-functor $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y) \to \Gamma(Y)$ identifies $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)$ with a sub-ω-category of $\Gamma(Y)$.

The vertical composition of n-cylinders can be described in the following way. Let $c = (x, y, \alpha)$ and let $d = (y, z, \beta)$ be two n-cylinders in Y (see paragraph 1.13). We define by induction on k such that $0 \leqslant k \leqslant n$ four $(k+1)$-cells a^ε_k and b^ε_k, with $\varepsilon = \pm$, as follows:

$$\begin{aligned} a^\varepsilon_k &= b^+_{k-1} *_{k-1} \cdots *_1 b^+_0 *_0 \alpha^\varepsilon_k, \\ b^\varepsilon_k &= \beta^\varepsilon_k *_0 a^-_0 *_1 \cdots *_{k-1} a^-_{k-1}. \end{aligned}$$

The vertical composition of d and c, denoted by $d *_v c$, is then given by the triple (x, z, γ), where

$$\gamma^\varepsilon_k = b^\varepsilon_k *_k a^\varepsilon_k.$$

Note that the unit of an n-cell x for the vertical composition is the n-cylinder (x, x, α), where $\alpha^-_k = 1_{s_k(x)}$ and $\alpha^+_k = 1_{t_k(x)}$.

Proposition 6.7. — *Let $c = (x, y, \alpha)$ be an invertible n-cylinder in an ω-category Y. Then c is invertible for the vertical composition with inverse (y, x, β) given by:*

$$\beta^\varepsilon_k = \bar\alpha^+_0 *_0 (\bar\alpha^+_1 *_1 \cdots *_{k-2} (\bar\alpha^+_{k-1} *_{k-1} \bar\alpha^\varepsilon_k *_{k-1} \bar\alpha^-_{k-1}) *_{k-2} \cdots *_1 \bar\alpha^-_1) *_0 \bar\alpha^-_0,$$

for $0 \leqslant k \leqslant n$ and $\varepsilon = \pm$, where $\bar\alpha^\varepsilon_l$ denotes the inverse of α^ε_l in Y.

Proof. — Let us first show that (y, x, β) is an n-cylinder. Let $1 \leqslant k \leqslant n$. We have to show that

$$t(\beta^\varepsilon_k) = x^\varepsilon_k *_0 \beta^-_0 *_1 \cdots *_{k-1} \beta^-_{k-1}.$$

For $i \leq k \leq n$, we set

$$\beta^\varepsilon_{k,i} = \bar\alpha^+_i *_i (\bar\alpha^+_{i+1} *_{i+1} \cdots *_{k-2} (\bar\alpha^+_{k-1} *_{k-1} \bar\alpha^\varepsilon_k *_{k-1} \bar\alpha^-_{k-1}) *_{k-2} \cdots *_{i+1} \bar\alpha^-_{i+1}) *_i \bar\alpha^-_i$$

and

$$u_i = (\alpha^+_{i-1} *_{i-1} \cdots *_1 \alpha^+_0 *_0 x^\varepsilon_k) *_i \beta^-_{i,i} *_{i+1} \cdots *_{k-1} \beta^-_{k-1,i}.$$

In particular, we have

$$\beta^{\varepsilon}_{k,0} = \beta^{\varepsilon}_{k}, \quad \bar{\alpha}^{+}_{i-1} *_i \beta^{\varepsilon}_{k,i} *_i \bar{\alpha}^{-}_{i-1} = \beta^{\varepsilon}_{k,i-1} \quad \text{and} \quad \beta^{\varepsilon}_{k,k} = \bar{\alpha}^{\varepsilon}_{k},$$

and

$$\begin{aligned} u_0 &= x^{\varepsilon}_k *_0 \beta^{-}_0 *_1 \cdots *_{k-1} \beta^{-}_{k-1}, \\ u_k &= \alpha^{+}_{k-1} *_{k-1} \cdots *_1 \alpha^{+}_0 *_0 x^{\varepsilon}_k = s(\alpha^{\varepsilon}_k). \end{aligned}$$

We have to show the equality $t(\beta^{\varepsilon}_k) = u_0$. More generally, we will prove by descending induction on i such that $0 \leq i \leq k$ that we have $t(\beta^{\varepsilon}_{k,i}) = u_i$. For $i = k$, we have

$$t(\beta^{\varepsilon}_{k,k}) = t(\bar{\alpha}^{\varepsilon}_k) = s(\alpha^{\varepsilon}_k) = u_k.$$

For $0 \leqslant i < k$, using the induction hypothesis, we have

$$\begin{aligned} t(\beta^{\varepsilon}_{k,i}) &= t(\alpha^{+}_i *_i \beta^{\varepsilon}_{k,i+1} *_i \alpha^{-}_i) \\ &= \alpha^{+}_i *_i u_{i+1} *_i \alpha^{-}_i \\ &= \bar{\alpha}^{+}_i \\ &\qquad *_i \big((\alpha^{+}_i *_i \cdots *_1 \alpha^{+}_0 *_0 x^{\varepsilon}_k) *_{i+1} \beta^{-}_{i+1,i+1} *_{i+2} \cdots *_{k-1} \beta^{-}_{k-1,i+1}\big) \\ &\qquad *_i \bar{\alpha}^{-}_i \\ &= \big(\bar{\alpha}^{+}_i *_i (\alpha^{+}_i *_i \cdots *_1 \alpha^{+}_0 *_0 x^{\varepsilon}_k) *_i \bar{\alpha}^{-}_i\big) \\ &\qquad *_{i+1} (\bar{\alpha}^{+}_i *_i \beta^{-}_{i+1,i+1} *_i \bar{\alpha}^{-}_i) *_{i+2} \cdots *_{k-1} (\bar{\alpha}^{+}_i *_i \beta^{-}_{k-1,i+1} \bar{\alpha}^{-}_i) \\ &= \big(\bar{\alpha}^{+}_i *_i (\alpha^{+}_i *_i \cdots *_1 \alpha^{+}_0 *_0 x^{\varepsilon}_k) *_i \bar{\alpha}^{-}_i\big) \\ &\qquad *_{i+1} \beta^{-}_{i+1,i} *_{i+2} \cdots *_{k-1} \beta^{-}_{k-1,i} \\ &= \big(\alpha^{+}_{i-1} *_{i-1} \cdots *_1 \alpha^{+}_0 *_0 x^{\varepsilon}_k\big) *_i \beta^{-}_{i,i} \\ &\qquad *_{i+1} \beta^{-}_{i+1,i} *_{i+2} \cdots *_{k-1} \beta^{-}_{k-1,i} \\ &= u_i. \end{aligned}$$

A similar calculation shows that

$$s(\beta^{\varepsilon}_k) = \beta^{+}_{k-1} *_{k-1} \cdots *_1 \beta^{+}_0 *_0 y^{\varepsilon}_k.$$

Let us now prove that (y, x, β) is an inverse of (x, y, α) for the vertical composition. Consider $(x, x, \gamma) = (y, x, \beta) *_v (x, y, \alpha)$. We have to prove that γ^{ε}_k is an identity for every $0 \leqslant k \leqslant n$ and $\varepsilon = \pm$. Recall that by definition (see paragraph 6.6), we have

$$\gamma^{\varepsilon}_k = b^{\varepsilon}_k *_k a^{\varepsilon}_k,$$

where

$$\begin{aligned} a^{\varepsilon}_k &= b^{+}_{k-1} *_{k-1} \cdots *_1 b^{+}_0 *_0 \alpha^{\varepsilon}_k, \\ b^{\varepsilon}_k &= \beta^{\varepsilon}_k *_0 a^{-}_0 *_1 \cdots *_{k-1} a^{-}_{k-1}. \end{aligned}$$

We will start by proving, by induction on k, that

$$\begin{aligned} a^{\varepsilon}_k &= \bar{\alpha}^{+}_0 *_0 \bar{\alpha}^{+}_1 *_1 \cdots *_{k-2} \bar{\alpha}^{+}_{k-1} *_{k-1} \alpha^{\varepsilon}_k, \\ b^{\varepsilon}_k &= \bar{\alpha}^{+}_0 *_0 \bar{\alpha}^{+}_1 *_1 \cdots *_{k-2} \bar{\alpha}^{+}_{k-1} *_{k-1} \bar{\alpha}^{\varepsilon}_k. \end{aligned}$$

For $k = 0$, the formulas boil down to the equalities $a^\varepsilon_0 = \alpha^\varepsilon_0$ and $b^\varepsilon_0 = \bar\alpha^\varepsilon_0$ which hold by definition. Suppose $k > 0$. For $0 \leqslant i \leq j \leq n$, we set

$$\begin{aligned} a^\varepsilon_{j,i} &= \bar\alpha^+_i *_i \bar\alpha^+_{i+1} *_{i+1} \cdots *_{j-2} \bar\alpha^+_{j-1} *_{j-1} \alpha^\varepsilon_j, \\ b^\varepsilon_{j,i} &= \bar\alpha^+_i *_i \bar\alpha^+_{i+1} *_{i+1} \cdots *_{j-2} \bar\alpha^+_{j-1} *_{j-1} \bar\alpha^\varepsilon_j. \end{aligned}$$

In particular, we have

$$a^\varepsilon_{j,j} = \bar\alpha^\varepsilon_j \quad \text{and} \quad b^\varepsilon_{j,j} = \bar\alpha^\varepsilon_j.$$

By induction hypothesis, for $i \leq j < k$, we have

$$b^\varepsilon_j = b^\varepsilon_{j,0}.$$

We thus have

$$\begin{aligned} a^\varepsilon_k &= b^+_{k-1} *_{k-1} \cdots *_1 b^+_0 *_0 \alpha^\varepsilon_k \\ &= b^+_{k-1,0} *_{k-1} \cdots *_1 b^+_{0,0} *_0 \alpha^\varepsilon_k \\ &= (\bar\alpha^+_0 *_0 b^+_{k-1,1}) *_{k-1} \cdots *_2 (\bar\alpha^+_0 *_0 b^+_{1,1}) *_1 \bar\alpha^+_0 *_0 \alpha^\varepsilon_k \\ &= \bar\alpha^+_0 *_0 (b^+_{k-1,1} *_{k-1} \cdots *_2 b^+_{1,1} *_1 \alpha^\varepsilon_k) \\ &= \cdots \\ &= \bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{i-2} \bar\alpha^+_{i-1} *_{i-1} (b^+_{k-1,i} *_{k-1} \cdots *_{i+1} b^+_{i,i} *_i \alpha^\varepsilon_k) \\ &= \cdots \\ &= \bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{k-2} \bar\alpha^+_{k-1} *_{k-1} \alpha^\varepsilon_k \end{aligned}$$

and

$$\begin{aligned} b^\varepsilon_k &= \beta^\varepsilon_k *_0 a^-_0 *_1 \cdots *_{k-1} a^-_{k-1} \\ &= \beta^\varepsilon_{k,0} *_0 a^-_{0,0} *_1 \cdots *_{k-1} a^-_{k-1,0} \\ &= \bar\alpha^+_0 *_0 \beta^\varepsilon_{k,1} *_0 \bar\alpha^-_0 *_0 \alpha^-_0 *_1 (\bar\alpha^+_0 *_0 a^-_{1,1}) *_2 \cdots *_{k-1} (\bar\alpha^+_0 *_0 a^-_{k-1,1}) \\ &= \bar\alpha^+_0 *_0 (\beta^\varepsilon_{k,1} *_1 a^-_{1,1} *_2 \cdots *_{k-1} a^-_{k-1,1}) \\ &= \cdots \\ &= \bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{i-2} \bar\alpha^+_{i-1} *_{i-1} (\beta^\varepsilon_{k,i} *_i a^-_{i,i} *_{i+1} \cdots *_{k-1} a^-_{k-1,i}) \\ &= \cdots \\ &= \bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{k-2} \bar\alpha^+_{k-1} *_{k-1} \beta^\varepsilon_{k,k} \\ &= \bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{k-2} \bar\alpha^+_{k-1} *_{k-1} \bar\alpha^\varepsilon_k, \end{aligned}$$

which ends the proof of the announced formulas. Finally, we get that

$$\begin{aligned} \gamma^\varepsilon_k &= b^\varepsilon_k *_k a^\varepsilon_k \\ &= \big(\bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{k-2} \bar\alpha^+_{k-1} *_{k-1} \bar\alpha^\varepsilon_k\big) \\ &\qquad *_k \big(\bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{k-2} \bar\alpha^+_{k-1} *_{k-1} \alpha^\varepsilon_k\big) \\ &= \bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{k-2} \bar\alpha^+_{k-1} *_{k-1} (\bar\alpha^\varepsilon_k *_k \alpha^\varepsilon_k) \\ &= 1_{\bar\alpha^+_0 *_0 \bar\alpha^+_1 *_1 \cdots *_{k-2} \bar\alpha^+_{k-1}}. \end{aligned}$$

This proves that $(y, x, \beta) *_v (x, y, \alpha)$ is indeed the identity. Similar calculations show that $(x, y, \alpha) *_v (y, x, \beta)$ is the identity as well, thereby ending the proof. $\square$

Proposition 6.8. — *Let Y be an ω-category. The ω-functor*

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y) \to \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, Y) = \Gamma(Y),$$

induces an isomorphism between $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)$ and $\Gamma_{\mathrm{inv}}(Y) \subset \Gamma(Y)$.

Proof. — We saw in paragraph 6.6 that this ω-functor is injective. It thus suffices to prove that its image is precisely $\Gamma_{\mathrm{inv}}(Y)$.

Let us first prove that the ω-functor lands into $\Gamma_{\mathrm{inv}}(Y)$. Consider an n-cell of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)$, seen as an ω-functor $c : \mathrm{I}_1 \otimes \mathrm{D}_n \to Y$. Let (x, y, α) be the associated cylinder. By definition, we have

$$\alpha_k^- = c((01) \otimes s_k(d)) \quad \text{and} \quad \alpha_k^+ = c((01) \otimes t_k(d)),$$

for $0 \leqslant k \leqslant n$, where (01) and d denote the principal cells of I_1 and D_n, respectively. As (01) is invertible in I_1, so is its tensor product with any cell by Proposition 6.4, and the α_k^ε are thus invertible. This proves that (x, y, α) is an invertible cylinder.

Reciprocally, if c is an invertible n-cylinder, then, by the previous proposition, the cylinder c is invertible for the vertical composition, and hence corresponds to an n-cell of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)$ (see paragraph 6.6), thereby proving the result. $\square$

Remark 6.9. — In particular, an n-cylinder is invertible in the sense of paragraph 6.5 if and only if it invertible for the vertical composition introduced in paragraph 6.6.

Proposition 6.10. — *An n-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$, seen by adjunction as an ω-functor $H : \mathrm{D}_n \otimes X \to Y$, is invertible if and only if, for every m-cell x of X, the $(n+m)$-cell $H(d \otimes x)$, where d denotes the principal cell of D_n, is invertible in Y.*

Proof. — Suppose that $H : \mathrm{D}_n \otimes X \to Y$ is invertible as an n-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$. By universal property of I_n and by adjunction, this means that H factors through $\mathrm{I}_n \otimes X$, so that $H(d \otimes x) = H'(d' \otimes x)$, where $H' : \mathrm{I}_n \otimes X \to Y$ and d' is the principal cell of I_n. As d' is invertible in I_n, so is $d' \otimes x$ by Proposition 6.4. This implies that $H(d \otimes x) = H'(d' \otimes x)$ is invertible, showing one implication.

Let us show the converse. We will argue by induction on $n \geqslant 1$. Suppose $n = 1$. The hypothesis implies that the associated ω-functor $k : X \to \Gamma(Y)$ factors through $\Gamma_{\mathrm{inv}}(Y) \subset \Gamma(Y)$. The result thus follows from the bijections

$$\begin{aligned}\mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X, \Gamma_{\mathrm{inv}}(Y)) &\simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X, \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{I}_1, Y)) \\ &\simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(\mathrm{I}_1, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)),\end{aligned}$$

the first bijection being a consequence of the previous proposition.

Suppose now that $n > 1$. Denote by $s : \mathrm{D}_{n-1} \otimes \mathrm{D}_1 \to \mathrm{D}_n$ the ω-functor of paragraph 4.7 and consider the ω-functor

$$H(s \otimes X) : \mathrm{D}_{n-1} \otimes \mathrm{D}_1 \otimes X \to Y.$$

Denote by e the principal cell of D_{n-1}. We will prove that, for every cell z of $\mathrm{D}_1 \otimes X$, the cell $H(s \otimes X)(e \otimes z)$ is invertible in Y. Since the ω-category $\mathrm{D}_1 \otimes X$ is generated by cells of the form $0 \otimes x$, $1 \otimes x$ and $(01) \otimes x$, where (01) denotes the principal cell of D_1 and x is a cell of X, it suffices to show that $H(s \otimes X)(e \otimes z)$, where z is one of these generators, is an invertible cell. Since $s(e \otimes \varepsilon) \otimes x$, for $\varepsilon = 0, 1$, is an identity by definition of s and the fact that e is of dimension at least 1, the cell $H(s \otimes X)(e \otimes \varepsilon \otimes x)$ is invertible. Furthermore, the cell

$$H(s \otimes X)(e \otimes (01) \otimes x) = H(s(e \otimes (01)) \otimes x) = H(d \otimes x)$$

is invertible by hypothesis on H. By induction hypothesis, this implies that $H(s \otimes X)$ is invertible as an $(n-1)$-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1 \otimes X, Y)$. This means that $H(s \otimes X)$ factors trough $\mathrm{I}_{n-1} \otimes (\mathrm{D}_1 \otimes X)$, so that we get an ω-functor $H' : \mathrm{I}_{n-1} \otimes (\mathrm{D}_1 \otimes X) \to Y$. Denote by d' and e' the principal cells of I_n and I_{n-1}, respectively. Using the fact that, by Proposition 6.4, the cell $e' \otimes (01)$ is invertible in $\mathrm{I}_{n-1} \otimes \mathrm{D}_1$, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{D}_n \otimes X \arrow[r, "\langle e \otimes (01) \rangle \otimes X"] \arrow[d, "\langle d' \rangle \otimes X"'] & \mathrm{D}_{n-1} \otimes \mathrm{D}_1 \otimes X \arrow[r, "s \otimes X"] \arrow[d, "\langle e \otimes (01) \rangle \otimes \mathrm{D}_1 \otimes X"'] & \mathrm{D}_n \otimes X \arrow[r, "H"] & Y \\
\mathrm{I}_n \otimes X \arrow[r, "\langle e' \otimes (01) \rangle \otimes X"] & \mathrm{I}_{n-1} \otimes \mathrm{D}_1 \otimes X \arrow[urr, "H'"'] & &
\end{tikzcd}.$$

Since $s\langle e \otimes (01) \rangle = 1_{\mathrm{D}_n}$, the composite of the three composable horizontal arrows of the diagram is $H : \mathrm{D}_n \otimes X \to Y$. This implies that H factors through $\mathrm{I}_n \otimes X$ and hence that it is invertible as a cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$. □

Remark 6.11. — In particular, an oplax transformation is invertible as a 1-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$ if and only it its components are invertible in Y.

Proposition 6.12. — *If X is an ω-category and Y is an (m, n)-category, then both $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}(X, Y)$ are (m, n)-categories.*

Proof. — Since

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(X, Y) \simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(X^{\mathrm{op}}, Y^{\mathrm{op}})^{\mathrm{op}}$$

(see paragraph 1.20) and (m, n)-categories are stable under the duality $Z \mapsto Z^{\mathrm{op}}$, it suffices to prove that $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$ is an (m, n)-category.

The fact that $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$ is an m-category is already known (see for instance [**2**, Proposition A.29]). Let us prove that this m-category is an (m, n)-category. Consider a k-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X, Y)$, with $k > n$, seen as an ω-functor $H : \mathrm{D}_k \otimes X \to Y$. Let d be the principal cell of D_k and let x be a cell of X. The cell $H(x \otimes d)$ is a cell of Y of dimension at least k. It is therefore invertible as Y is an (m,n)-category. Proposition 6.10 thus shows that H is invertible in $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(X,Y)$. $\square$

Theorem 6.13. — *The* Gray tensor product of (m,n)-categories

$$X \otimes_{m,n} Y = r(X \otimes Y),$$

where $r : \omega\text{-}\mathcal{C}at \to (m,n)\text{-}\mathcal{C}at$ denotes the left adjoint to the inclusion functor $(m,n)\text{-}\mathcal{C}at \hookrightarrow \omega\text{-}\mathcal{C}at$, defines a monoidal category structure on $(m,n)\text{-}\mathcal{C}at$, whose unit is the (m,n)-category D_0 and whose internal Hom *are* $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ *and* $\underline{\mathsf{Hom}}_{\mathrm{lax}}$.

Proof. — This follows from the previous proposition by Day's reflection theorem (Proposition 6.2). $\square$

Remark 6.14. — The case of m-categories was already proved in [**2**, Appendix A].

We now focus on the case of m-groupoids (that is, the case $n = 0$).

Proposition 6.15. — *If X and Y are two ω-groupoids, then $X \otimes Y$ is an ω-groupoid, so that*

$$X \otimes_{\omega,0} Y = X \otimes Y.$$

Proof. — The ω-category $X \otimes Y$ is generated by n-cells of the form $x \otimes y$, where x a k-cell of X and Y an l-cell of Y with $n = k + l$. If $n > 0$, then $k > 0$ or $l > 0$, so that x or y is invertible. It follows from Proposition 6.4 that $x \otimes y$ is invertible, thereby proving the result. $\square$

Remark 6.16. — Note that it is not true that the tensor product of an ω-groupoid X and an ω-category Y is an ω-groupoid in general, as the tensor product of a 0-cell of X and a non-invertible n-cell of Y is not invertible in $X \otimes Y$.

Lemma 6.17. — *The functor $X \mapsto X^{\mathrm{op}}$ is naturally isomorphic to the identity when restricted to the category $m\text{-}\mathcal{G}pd$ of m-groupoids.*

Proof. — Let X be an m-groupoid. Recall that for any n-cell x of X and any $i < n$, the cell x is invertible for the composition $*_i$ (see for instance [**3**, Proposition 1.3]). We will denote this inverse by $w_i(x)$. Note that for any $i, j < n$, we have $w_i(w_j(x)) = w_j(w_i(x))$. We define an ω-functor $\delta_X : X^{\mathrm{op}} \to X$ by setting $\delta_X(x) = w_1(w_3(\cdots w_k(x)\cdots))$, for any n-cell x of X, where k is the largest odd integer strictly smaller than n. The fact that this defines an ω-functor natural in X follows from a straightforward calculation. Clearly, the ω-functor δ_X is an isomorphism with inverse $\delta_{X^{\mathrm{op}}}$, thereby proving the result. $\square$

Proposition 6.18. — *The Gray tensor product on $m\text{-}\mathcal{G}pd$ is symmetric.*

Proof. — If X and Y are two m-groupoids, then, using the natural isomorphism of the previous lemma, we get an isomorphism

$$X \otimes_{m,0} Y = X \otimes Y \simeq (X \otimes Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes X^{\mathrm{op}} \simeq Y \otimes X = Y \otimes_{m,0} X.$$

One checks that this isomorphism defines a symmetry for the Gray tensor product. □

Let us come back to the general case $0 \leqslant n \leqslant m \leqslant \omega$. We will now show that the tensor product of (m,n)-categories is compatible with the so-called *folk model category structure on* (m,n)-$\mathcal{C}at$ that we now recall:

Theorem 6.19 **(Lafont–Métayer–Worytkiewicz, Ara–Métayer)**
The folk model category structure on ω-$\mathcal{C}at$ can be transferred along the adjunction

$$r : \omega\text{-}\mathcal{C}at \to (m,n)\text{-}\mathcal{C}at, \qquad (m,n)\text{-}\mathcal{C}at \hookrightarrow \omega\text{-}\mathcal{C}at.$$

In particular, we get a model category structure on (m,n)-$\mathcal{C}at$, whose weak equivalences are the folk equivalences between (m,n)-categories and which is cofibrantly generated by $r(I)$ and $r(J)$ (see paragraphs 1.6 and 1.7).

Proof. — The case of m-categories is [**15**, Theorem 5] and the case of (ω,n)-categories is [**3**, Theorem 3.19 and Remark 3.20]. Combining these two proofs, one easily gets the general case. □

The compatibility between the tensor product of (m,n)-categories and the folk model category structure on (m,n)-$\mathcal{C}at$ will follow formally from the following general statement:

Proposition 6.20. — *Let $\mathcal{M}$ be a biclosed monoidal model category, which is cofibrantly generated by sets I and J, and whose unit for the tensor product is cofibrant, and let $\mathcal{N} \subset \mathcal{M}$ be a reflective subcategory of $\mathcal{M}$. Denote by $r : \mathcal{M} \to \mathcal{N}$ the left adjoint to the inclusion functor. Suppose that*

(a) *$\mathcal{N} \subset \mathcal{M}$ satisfies the equivalent conditions of Day's reflection theorem (Proposition 6.2), so that*
$$X \otimes_{\mathcal{N}} Y = r(X \otimes Y)$$
defines a biclosed monoidal category structure on $\mathcal{N}$,

(b) *$\mathcal{N}$ is endowed with a model category structure cofibrantly generated by $r(I)$ and $r(J)$.*

Then $\mathcal{N}$ endowed with the tensor product $\otimes_{\mathcal{N}}$ is a monoidal model category.

Proof. — The hypothesis implies that r is a left Quillen functor. In particular, the unit of $\otimes_{\mathcal{N}}$ is cofibrant. By Lemma A.3, it suffices to show that if i is in I and j is in I, then $r(i) \otimes'_{\mathcal{N}} r(j)$ is a cofibration of $\mathcal{N}$, and that if either i is in I and j is in J, or i is in J and j is in I, then $r(i) \otimes'_{\mathcal{N}} r(j)$ is a trivial cofibration. Using the definition of $\otimes_{\mathcal{N}}$, the natural isomorphism $r(X \otimes Y) \simeq r(r(X) \otimes r(Y))$ and the fact that r preserves pushouts, we get that $r(i) \otimes'_{\mathcal{N}} r(j)$ can be identified with $r(i \otimes' j)$.

The result thus follows from the pushout-product axiom in $\mathcal{M}$ and the fact that r is a left Quillen functor. $\square$

Theorem 6.21. — *The folk model category structure on (m,n)-$\mathcal{C}at$ is monoidal for the Gray tensor product of (m,n)-categories.*

Proof. — This follows from the previous proposition, whose hypothesis are fulfilled by Proposition 6.12 and Theorem 6.19. $\square$

Remark 6.22. — In [**13**], slightly corrected by [**14**], Lack proves that the folk model category structure on 2-$\mathcal{C}at$ is monoidal for the *pseudo* Gray tensor product. This is different from the result we get from the previous theorem in the case $m = 2$ and $n = 2$, which deals with the *oplax* Gray tensor product.

Proposition 6.23. — *The folk model category structure on (m,n)-$\mathcal{C}at$ satisfies the two following properties:*

(a) *Transfinite compositions of pushouts of tensor products of an object (on the left) and a folk trivial cofibration are folk weak equivalences.*
(b) *Transfinite compositions of pushouts of tensor products of a folk trivial cofibration and an object (on the right) are folk weak equivalences.*

Proof. — The second assertion can be deduced from the first one using the duality $X \mapsto X^{\mathrm{op}}$. As for the first one, using the fact that the tensor product of (m,n)-categories is biclosed and hence commutes with colimits in each variable, it suffices to consider transfinite compositions of pushouts of tensor products of an object and an element of $r(J)$. As the functor r commutes with colimits, such a transfinite composition is of the form $r(f)$, where f is a transfinite composition of pushouts of tensor products of an object and an element of J. By Remark 5.11, such an f is a transfinite composition of J_1-transformation retracts. As folk weak equivalences are stable under transfinite compositions, it suffices to show that r sends J_1-transformation retracts to weak equivalences. But if $h : \mathrm{J}_1 \otimes X \to Y$ is a J_1-transformation from an ω-functor $u : X \to Y$ to an ω-functor $v : X \to Y$, then by precomposing $r(h) : r(\mathrm{J}_1 \otimes X) \to r(Y)$ by the natural ω-functor

$$\mathrm{J}_1 \otimes r(X) \to r(\mathrm{J}_1) \otimes r(X) \to r(r(\mathrm{J}_1) \otimes r(X)) \simeq r(\mathrm{J}_1 \otimes X),$$

one gets an ω-functor $\mathrm{J}_1 \otimes r(X) \to r(Y)$ defining a J_1-transformation from $r(u)$ to $r(v)$. This proves that r sends J_1-transformation retracts to J_1-transformation retracts, hence the result by Proposition 4.13. $\square$

Remark 6.24. — In the case $n = 0$, in which the tensor product is symmetric, the previous proposition asserts that the so-called *monoid axiom* of Schwede and Shipley [**21**] holds in m-$\mathcal{G}pd$.

Corollary 6.25. — *Let us endow m-$\mathcal{G}pd$ with the Gray tensor product.*

(a) *If $\mathcal{P}$ is a non-symmetric $(m\text{-}\mathcal{G}pd)$-operad, then the category of $\mathcal{P}$-algebras in m-groupoids is endowed with a right proper combinatorial model category structure whose weak equivalences* (resp. *fibrations*) *are the morphisms whose underlying m-functor is a folk weak equivalence* (resp. *a folk fibration*) *of m-groupoids.*

(b) *If $\mathcal{P}$ is a non-symmetric $(m\text{-}\mathcal{G}pd)$-operad such that $\mathcal{P}(n)$ is a cofibrant m-groupoid for every $n \geqslant 0$, and if A is a cofibrant $\mathcal{P}$-algebra in $m\text{-}\mathcal{G}pd$, then the underlying m-groupoid of A is cofibrant.*

Proof. — Using the previous remark, the first point is [**9**, Theorem 1.2] (or also [**19**, Theorem 1.3], taking $\mathcal{V} = \mathcal{C} = m\text{-}\mathcal{G}pd$).

Let us prove the second point using results and terminology from [**4**]. Let $\widetilde{\mathcal{P}}$ be the free symmetric operad on the non-symmetric operad $\mathcal{P}$. We have $\widetilde{\mathcal{P}}(n) = \Sigma_n \times \mathcal{P}(n)$, where Σ_n denotes the symmetric group, the action of Σ_n on $\widetilde{\mathcal{P}}(n)$ being the obvious one. Since $\mathcal{P}(n)$ is cofibrant by hypothesis, this implies that, in the terminology of [**4**], the operad $\widetilde{\mathcal{P}}$ is Σ-cofibrant. Moreover, the category of $\widetilde{\mathcal{P}}$-algebras is isomorphic to the category of $\mathcal{P}$-algebras via a functor constant on the underlying object and, by the first point, this category is thus endowed with a model category structure compatible with the forgetful functor to $m\text{-}\mathcal{G}pd$. In the terminology of [**4**], this means that $\widetilde{\mathcal{P}}$ is an admissible operad. The result thus follows from [**4**, Corollary 5.5]. $\square$

Remark 6.26. — In particular, the first point applied to the operad of monoids (seen as an $(m\text{-}\mathcal{G}pd)$-operad by using the inclusion functor of sets into m-groupoids) gives a model category structure on the category of monoids in the category of m-groupoids endowed with the Gray tensor product.

Remark 6.27. — The monoid axiom implies many other interesting properties of the homotopy theory of operads and their algebras. Another important setting to obtain these kinds of results has been introduced by Berger and Moerdijk in [**4**]. One easily checks that the folk model category structure on $m\text{-}\mathcal{G}pd$, equipped with the Gray tensor product, satisfies the hypothesis of [**4**, Theorem 3.1], the "Hopf interval" being simply the m-groupoid I_1.

7. The folk model category structure is monoidal for the join

In this section, we will recall the definition of the join of ω-categories, introduced by the first author and Maltsiniotis in [**2**], and we will prove that the resulting monoidal category structure is compatible with the folk model category structure.

The strategy to define the join is similar to the one for the Gray tensor product. In particular, we start by defining the join at the level of augmented directed complexes.

7.1. — The *join* $K \star L$ of two augmented directed complexes K and L is defined in the following way:

– For $n \geqslant 0$, we have
$$(K \star L)_n = \bigoplus_{\substack{i+1+j=n \\ i \geqslant -1, j \geqslant -1}} K_i \otimes L_j,$$
where by convention $K_{-1} = \mathbb{Z}$ and $L_{-1} = \mathbb{Z}$. The positive generator of K_{-1} or L_{-1} will be denoted by $\varnothing$. If x is in K_i and y is in K_j, we will denote by $x \star y$ the element $x \otimes y$ seen as an $(i+1+j)$-chain of $K \star L$.
– For x in K_i and y in K_j with $i+1+j \geqslant 1$, we have
$$d(x \star y) = dx \star y + (-1)^{i+1} x \star dy,$$
where by convention $dz = e(z)\varnothing$ if the degree of z is 0, and $dz = 0$ if the degree of z is -1.
– For x in K_0 and y in L_0, we have
$$e(x \star \varnothing) = e(x) \quad \text{and} \quad e(\varnothing \star y) = e(y).$$
– The submonoid $(K \star L)^*_n$ is defined to be generated by the subset
$$\bigoplus_{\substack{i+1+j=n \\ i \geqslant -1, j \geqslant -1}} K^*_i \otimes L^*_j$$
of $(K \star L)_n$.

The join defines a (non-symmetric) monoidal category structure on the category of augmented directed complexes. Moreover, the first author and Maltsiniotis proved (see [**2**, Corollary 6.21]) that this monoidal category structure restricts to the full subcategory of strong Steiner complexes.

Theorem 7.2 **(Ara–Maltsiniotis)**. — *There exists a unique, up to unique isomorphism, locally biclosed monoidal category structure (see paragraph A.12) on ω-$\mathcal{C}at$ making the functor $\nu_{|\mathcal{S}t_{\mathrm{f}}} : \mathcal{S}t_{\mathrm{f}} \to \omega\text{-}\mathcal{C}at$ a monoidal functor, where $\mathcal{S}t_{\mathrm{f}}$ is endowed with the monoidal category structure given by the join.*

Proof. — This is [**2**, Theorem 6.29]. ☐

7.3. — We define the *join* of ω-categories to be the monoidal product given by the previous theorem. If X and Y are two ω-categories, their join will be denoted by $X \star Y$. Explicitly, one has
$$X \star Y = \varinjlim_{\substack{\nu(K) \to X,\, K \in \mathcal{S}t_{\mathrm{f}} \\ \nu(L) \to Y,\, L \in \mathcal{S}t_{\mathrm{f}}}} \nu(K \star L).$$
The unit of the join is the empty ω-category. As a consequence, if X and Y are two ω-categories, we get canonical ω-functors $\iota_1 : X \to X \star Y$ and $\iota_2 : Y \to X \star Y$.

The fact that the join is locally biclosed means that the functors
$$\begin{aligned} \omega\text{-}\mathcal{C}at &\to {}_{X}\backslash\omega\text{-}\mathcal{C}at \\ Y &\mapsto (X \star Y, \iota_1 : X \to X \star Y) \end{aligned}$$

and

$$
\begin{aligned}
\omega\text{-}\mathcal{C}at &\to {}_{Y}\backslash\omega\text{-}\mathcal{C}at \\
X &\mapsto (X \star Y, \iota_2 : Y \to X \star Y)
\end{aligned}
$$

admit right adjoints. We thus get pairs of adjoint functors

$$
\begin{aligned}
\omega\text{-}\mathcal{C}at &\to {}_{X}\backslash\omega\text{-}\mathcal{C}at, & {}_{X}\backslash\omega\text{-}\mathcal{C}at &\to \omega\text{-}\mathcal{C}at \\
Y &\mapsto (X \star Y, \iota_1) & (Z, X \xrightarrow{u} Z) &\mapsto {}_{u}\backslash Z
\end{aligned}
$$

and

$$
\begin{aligned}
\omega\text{-}\mathcal{C}at &\to {}_{Y}\backslash\omega\text{-}\mathcal{C}at, & {}_{Y}\backslash\omega\text{-}\mathcal{C}at &\to \omega\text{-}\mathcal{C}at, \\
X &\mapsto (X \star Y, \iota_2) & (Z, Y \xrightarrow{v} Z) &\mapsto Z \overset{\mathrm{co}}{/} v
\end{aligned}
$$

so that, if X and Y are ω-categories and $u : X \to Z$ and $v : Y \to Z$ are ω-functors, we have natural bijections

$$
\begin{aligned}
\mathsf{Hom}_{{}_{X}\backslash\omega\text{-}\mathcal{C}at}((X \star Y, \iota_1), (Z, u)) &\simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(Y, {}_{u}\backslash Z), \\
\mathsf{Hom}_{{}_{Y}\backslash\omega\text{-}\mathcal{C}at}((X \star Y, \iota_2), (Z, v)) &\simeq \mathsf{Hom}_{\omega\text{-}\mathcal{C}at}(X, Z \overset{\mathrm{co}}{/} v).
\end{aligned}
$$

(See [**2**, Remark 6.37] for the reason for the decoration "co" in $Z \overset{\mathrm{co}}{/} v$.)

One important consequence of the existence of these adjoints is that the join commutes with connected colimits in each variable.

Examples 7.4. — Here are some examples of joints of ω-categories:

$$
\mathrm{D}_0 \star \mathrm{D}_0 = \bullet \longrightarrow \bullet = \mathrm{D}_1 \quad , \qquad
\mathrm{D}_0 \star \mathrm{D}_1 = \begin{tikzcd}[row sep=small]
 & \bullet \arrow[dd] \\
\bullet \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Uparrow"] & {} \\
 & \bullet
\end{tikzcd} \quad ,
$$

$$
\mathrm{D}_1 \star \mathrm{D}_1 = \begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[r, phantom, shift right=3, "\Downarrow"] & \bullet \\
\bullet \arrow[r] \arrow[ur, phantom, "\Swarrow"] & \bullet \arrow[u]
\end{tikzcd}
\Rrightarrow
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[r, phantom, shift right=3, "\Downarrow"] & \bullet \\
\bullet \arrow[r] \arrow[ur] \arrow[ur, phantom, shift right=3, "\Searrow"] & \bullet \arrow[u]
\end{tikzcd} \quad .
$$

We now begin to prove that the join is compatible with the folk model category structure.

Proposition 7.5. — *If*

$$i : X \to Y \quad \textit{and} \quad j : Z \to T$$

are two folk cofibrations, then the ω-functor

$$i \star' j : Y \star Z \amalg_{X \star Z} X \star T \to Y \star T$$

is also a folk cofibration.

Proof. — The proof is essentially the same as the one of Theorem 3.9. It is immediate that if K and L are two augmented directed complexes with basis (that we denote by B_K and B_L, respectively, following the notation introduced in paragraph 3.1), then $K \star L$ (see paragraph 7.1) is an augmented directed complex with basis

$$B_{K\star L} = \{x \star y \mid x \in B_K,\, y \in B_L\} \cup \{x \star \varnothing \mid x \in B_K\} \cup \{\varnothing \star y \mid y \in B_L\}.$$

From this, one deduces as in Proposition 3.5 that if $i : K \to L$ and $j : M \to N$ are two rigid monomorphisms between augmented directed complexes with basis, then the morphism

$$i \star' j : L \star M \amalg_{K \star M} K \star N \to L \star N$$

is a rigid monomorphism between augmented directed complexes with basis which identifies $L \star M \amalg_{K\star M} K \star N$ with the subcomplex generated by

$$B_L \star B_M \cup B_K \star B_N \cup B_L \star \{\varnothing\} \cup \{\varnothing\} \star B_N.$$

One then deduces, as in Proposition 3.7, that if K, L, M and N are assumed to be strong Steiner complexes, then the ω-functor

$$\nu(i) \star' \nu(j) : \nu(L) \star \nu(M) \amalg_{\nu(K)\star\nu(M)} \nu(K) \star \nu(N) \to \nu(L) \star \nu(N)$$

is a folk cofibration. To do so, one needs the fact that rigid monomorphisms and strong Steiner complexes are stable under join (see [**2**, Proposition 6.17 and Corollary 6.21]) and that the functor $\nu_{|\mathcal{S}t_{\mathrm{f}}} : \mathcal{S}t_{\mathrm{f}} \to \omega\text{-}\mathcal{C}at$ is monoidal for the join (see [**2**, Theorem 6.29]). The result then follows from the fact that generating cofibrations are of the form $\nu(i)$ for i a rigid monomorphism between strong Steiner complexes (see Proposition 3.8). $\square$

Corollary 7.6. — *The join of two cofibrant ω-categories is a cofibrant ω-category.*

Proof. — This follows immediately from the previous proposition. $\square$

To end the proof of the compatibility of the join with the folk model category structure, we will apply Lemma 5.1. The only non-trivial remaining hypothesis to be checked is H5). To do so, we will prove that the class of J_1-transformations is stable by taking the join by an object on the right (a fact which was trivial for the tensor product) by showing that, for every ω-category T, the functor $- \star T$ can be endowed with what is called a tensorial strength for the Gray tensor product. We start by defining this tensorial strength at the level of augmented directed complexes.

7.7. — Let K, L and M be three augmented directed complexes. We define a morphism

$$\sigma : K \otimes (L \star M) \to (K \otimes L) \star M,$$

natural in K, L and M, in the following way. For $n \geqslant 0$, we set

$$\sigma_n(x \otimes (y \star z)) = \begin{cases} e(x)(y \star z) & \text{if } |y| = -1, \\ (x \otimes y) \star z & \text{if } |y| \geqslant 0, \end{cases}$$

where by convention $e(x) = 0$ if x is not of degree 0.

Proposition 7.8. — *The morphisms σ_n define a morphism of augmented directed complexes $\sigma : K \otimes (L \star M) \to (K \otimes L) \star M$.*

Proof. — It is immediate that σ_n respects positive elements. Let us prove that σ_0 is compatible with the augmentations. If the degree of $x \otimes (y \star z)$ is 0 then $|x| = 0$ and

- either $|y| = -1$ and $|z| = 0$, in which case we have
$$e(\sigma_0(x \otimes (y \star z))) = e(e(x)(y \star z)) = e(x)e(y \star z) = e(x \otimes (y \star z)),$$
- or $|y| = 0$ and $|z| = -1$, in which case we can assume that $z = \varnothing$ and we have
$$\begin{aligned} e(\sigma_0(x \otimes (y \star \varnothing))) &= e((x \otimes y) \star \varnothing) = e(x \otimes y) \\ &= e(x)e(y) = e(x)e(y \star \varnothing) = e(x \otimes (y \star \varnothing)). \end{aligned}$$

Suppose now that the degree n of $x \otimes (y \star z)$ is at least 1 and let us prove that
$$\sigma_{n-1}d(x \otimes (y \star z)) = d\sigma_n(x \otimes (y \star z)).$$

We will freely use the conventions for the differentials of tensors and joins introduced in paragraphs 2.9 and 7.1. We distinguish four cases:

- If $|y| = -1$, then we have
$$\begin{aligned} \sigma_{n-1}d(x \otimes (y \star z)) &= \sigma_{n-1}\big(dx \otimes (y \star z) + (-1)^{|x|} x \otimes d(y \star z)\big) \\ &= e(dx)(y \star z) + (-1)^{|x|} e(x) d(y \star z) \\ &\qquad \text{(being careful with the case } |z| = 0) \\ &= e(x)d(y \star z) \\ &\qquad \text{(as } ed = 0 \text{ and } e(x) = 0 \text{ if } |x| \neq 0) \\ &= d(e(x)(y \star z)) \\ &= d\sigma_n(x \otimes (y \star z)). \end{aligned}$$
- If $|y| = 0$ and $|z| = -1$, then we have
$$\begin{aligned} \sigma_{n-1}d(x \otimes (y \star z)) &= \sigma_{n-1}(dx \otimes (y \star z)) \\ &= (dx \otimes y) \star z \\ &= d(x \otimes y) \star z \\ &= d((x \otimes y) \star z) \\ &= d\sigma_n(x \otimes (y \star z)). \end{aligned}$$
- If $|y| = 0$ and $|z| \geqslant 0$, then we have
$$\begin{aligned} &\sigma_{n-1}d(x \otimes (y \star z)) \\ &\quad = \sigma_{n-1}\big(dx \otimes (y \star z) + (-1)^{|x|} x \otimes d(y \star z)\big) \\ &\quad = \sigma_{n-1}\big(dx \otimes (y \star z) + (-1)^{|x|} x \otimes (e(y)\varnothing \star z) + (-1)^{|x|+|y|+1} x \otimes (y \star dz)\big) \end{aligned}$$

$$= (dx \otimes y) \star z + (-1)^{|x|} e(x)e(y)(\varnothing \star z) + (-1)^{|x|+|y|+1}(x \otimes y) \star dz$$

(being careful with the case $|z| = 0$)

$$= (dx \otimes y) \star z + \big(e(x \otimes y)\varnothing \star z\big) + (-1)^{|x\otimes y|+1}(x \otimes y) \star dz$$

(as the second term is null if $|x| \neq 0$)

$$= d(x \otimes y) \star z + (-1)^{|x\otimes y|+1}(x \otimes y) \star dz$$

(distinguishing the cases $|x| = 0$ and $|x| \neq 0$)

$$= d((x \otimes y) \star z)$$

$$= d\sigma_n(x \otimes (y \star z)).$$

– If $|y| \geqslant 1$, then we have

$$\sigma_{n-1} d(x \otimes (y \star z))$$

$$= \sigma_{n-1}\big(dx \otimes (y \star z) + (-1)^{|x|} x \otimes d(y \star z)\big)$$

$$= \sigma_{n-1}\big(dx \otimes (y \star z) + (-1)^{|x|} x \otimes (dy \star z) + (-1)^{|x|+|y|+1} x \otimes (y \star dz)\big)$$

$$= (dx \otimes y) \star z + (-1)^{|x|}(x \otimes dy) \star z + (-1)^{|x|+|y|+1}(x \otimes y) \star dz$$

(being careful with the cases $|z| = -1$ and $|z| = 0$)

$$= d(x \otimes y) \star z + (-1)^{|x\otimes y|+1}(x \otimes y) \star dz$$

$$= d((x \otimes y) \star z)$$

$$= d\sigma_n(x \otimes (y \star z)),$$

thereby proving the result. ☐

Proposition 7.9. — *For any augmented directed complex T, the morphism σ of the previous proposition defines a tensorial strength for the tensor product on the functor*

$$\begin{aligned} \mathcal{C}_{\mathrm{ad}} &\to \mathcal{C}_{\mathrm{ad}} \\ K &\mapsto K \star T, \end{aligned}$$

meaning that, for every augmented directed complexes K, L and M, the triangles

$$\begin{tikzcd} K \otimes L \otimes (M \star T) \arrow[rr, "K\otimes s"] \arrow[dr, "s"'] & & K \otimes ((L \otimes M) \star T) \arrow[dl, "s"] \\ & (K \otimes L \otimes M) \star T & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{Z} \otimes (K \star T) \arrow[rr, "s"] \arrow[dr, "\lambda"'] & & (\mathbb{Z} \otimes K) \star T \arrow[dl, "\lambda\star T"] \\ & K \star T & \end{tikzcd} ,$$

where we have neglected the associativity constraints and λ denotes the left unit constraint, are commutative.

Proof. — This follows from direct calculations. $\square$

7.10. — Let X, Y and Z be three ω-categories. We define an ω-functor

$$s : X \otimes (Y \star Z) \to (X \otimes Y) \star Z,$$

natural in X, Y and Z, in the following way. First, recall that any ω-category T is a canonical colimit of ω-categories associated to strong Steiner complexes in the sense that we have a canonical isomorphism

$$T \simeq \varinjlim_{\nu(K)\to T,\, T\in\mathcal{S}t_{\mathrm{f}}} \nu(K).$$

(This follows from the fact that $\nu(\mathcal{S}t_{\mathrm{f}})$ contains Joyal's category Θ which is dense in ω-$\mathcal{C}at$, see also [**22**, Theorem 7.1].) This colimit is connected as the null augmented directed complex is an initial object of the category of strong Steiner complexes. As by Theorems 2.10 and 7.2, both the Gray tensor product and the join commute with connected colimits in each variable and are compatible with the functor $\nu_{|\mathcal{S}t_{\mathrm{f}}} : \mathcal{S}t_{\mathrm{f}} \to \omega\text{-}\mathcal{C}at$, we get canonical isomorphisms

$$X \otimes (Y \star Z) \simeq \varinjlim_{\substack{\nu(K)\to X,\, K\in\mathcal{S}t_{\mathrm{f}}\\ \nu(L)\to Y,\, L\in\mathcal{S}t_{\mathrm{f}}\\ \nu(M)\to Z,\, M\in\mathcal{S}t_{\mathrm{f}}}} \nu(K \otimes (L \star M)),$$

$$(X \otimes Y) \star Z \simeq \varinjlim_{\substack{\nu(K)\to X,\, K\in\mathcal{S}t_{\mathrm{f}}\\ \nu(L)\to Y,\, L\in\mathcal{S}t_{\mathrm{f}}\\ \nu(M)\to Z,\, M\in\mathcal{S}t_{\mathrm{f}}}} \nu((K \otimes L) \star M).$$

We thus obtain our ω-functor $s : X \otimes (Y \star Z) \to (X \otimes Y) \star Z$ by taking the colimit over K, L and M of the ω-functors

$$\nu(\sigma) : \nu(K \otimes (L \star M)) \to \nu((K \otimes L) \star M),$$

where $\sigma : K \otimes (L \star M) \to (K \otimes L) \star M$ is the morphism of Proposition 7.8.

Proposition 7.11. — *For any ω-category T, the ω-functor s of the previous paragraph defines a tensorial strength for the Gray tensor product on the functor*

$$\begin{aligned} \omega\text{-}\mathcal{C}at &\to \omega\text{-}\mathcal{C}at \\ X &\mapsto X \star T, \end{aligned}$$

meaning that, for every ω-categories X, Y and Z, the triangles

$$\begin{tikzcd} X \otimes Y \otimes (Z \star T) \arrow[rr, "X\otimes s"] \arrow[dr, "s"'] & & X \otimes ((Y \otimes Z) \star T) \arrow[dl, "s"] \\ & (X \otimes Y \otimes Z) \star T & \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{D}_0 \otimes (X \star T) \arrow[rr, "s"] \arrow[dr, "\lambda"'] & & (\mathrm{D}_0 \otimes X) \star T \arrow[dl, "\lambda \star T"] \\
& X \star T &
\end{tikzcd},$$

where we have neglected the associativity constraints and λ *denotes the left unit constraint, are commutative.*

Proof. — Using the same arguments as for the definition of s in terms of σ, we get that these two triangles are colimits of the image by ν of triangles as in Proposition 7.9. The result thus follows from this proposition, which asserts that these triangles are commutative. □

Proposition 7.12. — *Let* $f, g : X \to Y$ *be two* ω*-functors. If* $h : \mathrm{J}_1 \otimes X \to Y$ *is a* J_1*-transformation from* f *to* g*, then, for any* ω*-category* Z*, the* ω*-functor*

$$\mathrm{J}_1 \otimes (X \star Z) \xrightarrow{s} (\mathrm{J}_1 \otimes X) \star Z \xrightarrow{h \star Z} Y \star Z$$

defines a J_1*-transformation from* $f \star Z$ *to* $g \star Z$*.*

Proof. — This follows from the commutativity of the diagram

$$\begin{tikzcd}
X \star Z \arrow[d, "\simeq"'] \arrow[dr, "\simeq"] \arrow[ddrr, bend left, "f \star Z"] & & \\
\mathrm{D}_0 \otimes (X \star Z) \arrow[r, "s"] \arrow[d, "\langle 0 \rangle \otimes (X \star Z)"'] & (\mathrm{D}_0 \otimes X) \star Z \arrow[d, "(\langle 0 \rangle \otimes X) \star Z"] & \\
\mathrm{J}_1 \otimes (X \star Z) \arrow[r, "s"] & (\mathrm{J}_1 \otimes X) \star Z \arrow[r, "h \star Z"] & Y \star Z \\
\mathrm{D}_0 \otimes (X \star Z) \arrow[r, "s"] \arrow[u, "\langle 1 \rangle \otimes (X \star Z)"] & (\mathrm{D}_0 \otimes X) \star Z \arrow[u, "(\langle 1 \rangle \otimes X) \star Z"'] & \\
X \star Z \arrow[u, "\simeq"] \arrow[ur, "\simeq"'] \arrow[uurr, bend right, "g \star Z"'] & &
\end{tikzcd},$$

the two squares in the middle of the diagram being commutative by naturality of s, the two triangles by the previous proposition, and the two other small diagrams by hypothesis on h. □

Remark 7.13. — The analogous statement for oplax transformations, obtained by replacing J_1 by D_1, is true as well, the proof applying *mutatis mutandis*.

Theorem 7.14. — *The folk model category structure on* ω*-*$\mathcal{C}at$ *is monoidal for the join.*

Proof. — We apply Lemma 5.1. The hypothesis H1) and H2) are true for the folk model category structure and the hypothesis H3) is true for any locally biclosed monoidal category, as the initial object is the tensor unit. The hypothesis H4) follows from Proposition 7.5.

It remains to prove H5). Let i be a generating trivial cofibration and let Z be an ω-category. As seen in the proof of Proposition 5.4, the ω-functor i is a J_1-deformation retract. The previous proposition thus implies that $i \star Z$ is also a J_1-deformation retract, and therefore a weak equivalence by Proposition 4.13. The fact that $Z \star i$ is also a weak equivalence follows from a duality argument, as in the proof of Corollary 5.5, using the canonical natural isomorphism

$$(X \star Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \star X^{\mathrm{op}}$$

(see [**2**, Proposition 6.35]). This ends the proof of H5) and hence of the theorem. □

The statement analogous to Proposition 5.9 holds as well, the proof being a direct adaption:

Proposition 7.15. —
(a) *Transfinite compositions of pushouts of join of an object (on the left) and a folk trivial cofibration are folk weak equivalences.*
(b) *Transfinite compositions of pushouts of join of a folk trivial cofibration and an object (on the right) are folk weak equivalences.*

Remark 7.16. — In particular, the join of a folk trivial cofibration by an object (on the left or on the right) is a folk weak equivalence.

Finally, as for the Gray tensor product, the join of ω-categories induces a join of m-categories, which is compatible with the folk model category structure on m-$\mathcal{C}at$:

***Theorem 7.17* (Ara–Maltsiniotis)**. — *Let $m \geqslant 0$. The* join of m-categories

$$X \star_m Y = r(X \star Y),$$

where $r : \omega\text{-}\mathcal{C}at \to m\text{-}\mathcal{C}at$ denotes the left adjoint to the inclusion functor $m\text{-}\mathcal{C}at \hookrightarrow \omega\text{-}\mathcal{C}at$, defines a locally biclosed monoidal category structure on $m\text{-}\mathcal{C}at$.

Proof. — This is the main result of [**2**, Chapter 8]. □

Theorem 7.18. — *The folk model category structure on m-$\mathcal{C}at$ is monoidal for the join of m-categories.*

Proof. — This follows from a straightforward adaption of the proof of Proposition 6.20, which only uses connected colimits, with whom any locally biclosed monoidal tensor commutes. □

Appendix A
Monoidal model categories and derived tensor products

In this appendix, we recall classical results on biclosed monoidal model categories and extend them to locally biclosed monoidal model categories. In particular, we will get that the “local internal $\underline{\mathsf{Hom}}$” of the join, the so-called generalized slices, can be right-derived as functors of two variables.

A.1. — A *monoidal model category* is a model category whose underlying category is endowed with a monoidal category structure satisfying the following compatibility axioms:

M1) the tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfies the *pushout-product axiom*: if $i : A \to B$ and $j : C \to D$ are two cofibrations, then the morphism

$$i \otimes' j : B \otimes C \amalg_{A\otimes C} A \otimes D \to B \otimes D,$$

induced by the commutative square

$$\begin{tikzcd} A \otimes C \arrow[r, "A\otimes j"] \arrow[d, "i\otimes C"'] & A \otimes D \arrow[d, "i\otimes D"] \\ B \otimes C \arrow[r, "B\otimes j"'] & B \otimes D \end{tikzcd},$$

is a cofibration. Moreover, if either i or j is a trivial cofibration, then so is $i \otimes' j$.

M2) the tensor product satisfies the *unit axiom*: for every cofibrant replacement $p : QI \overset{\sim}{\twoheadrightarrow} I$ of the tensor unit and every cofibrant object A, both $p \otimes A : QI \otimes A \to I \otimes A$ and $A \otimes p : A \otimes QI \to A \otimes I$ are weak equivalences.

Remark A.2. — The pushout-product axiom implies that, if the cofibrations (resp. the trivial cofibrations) are stable by tensoring by the initial object $\varnothing$, then they are stable by tensoring by any cofibrant object X. In particular, if this condition for trivial cofibrations is satisfied and the tensor unit I is cofibrant, then, by Ken Brown’s lemma, the pushout-product axiom implies the unit axiom.

Lemma A.3. — *Let $\mathcal{M}$ be a cofibrantly generated model category with sets of generating cofibrations I and of generating trivial cofibrations J and let $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be a functor which commutes with pushouts and transfinite compositions in each variable. Then for $\otimes$ to satisfy the pushout-product axioms, it suffices that it holds for cofibrations in I and trivial cofibrations in J.*

Proof. — See for instance (the proof of) [**18**, Lemma 4.1.4]. □

Remark A.4. — More precisely, in the situation of the previous lemma, each of the three conditions appearing in the pushout-product axiom can be checked on “generators”.

Proposition A.5. — *If $\mathcal{M}$ is a monoidal model category having the additional property that trivial cofibrations are stable by tensoring by the initial object, then the tensor product admits a total left derived functor $\otimes^{\mathbb{L}} : \mathrm{Ho}(\mathcal{M}) \times \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ and this derived tensor product defines a monoidal category structure on* $\mathrm{Ho}(\mathcal{M})$*.*

Proof. — It follows from Remark A.2 that the tensor product of two trivial cofibrations between cofibrant objects is a weak equivalence. By Ken Brown's lemma, this implies that the tensor product preserves weak equivalences between cofibrant objects and hence, by a classical result of Quillen [**20**, I.4, Proposition 1], that its total left derived functor $\otimes^{\mathbb{L}}$ exists. Checking that $\otimes^{\mathbb{L}}$ indeed defines a monoidal category structure on $\mathrm{Ho}(\mathcal{M})$ is not difficult (see the proof of [**10**, Theorem 4.3.2]). □

A.6. — Recall that a monoidal category $\mathcal{C}$ is said to be *biclosed* if, for every object X of $\mathcal{C}$, the functor

$$\begin{aligned} \mathcal{C} &\to \mathcal{C} \\ Y &\mapsto X \otimes Y \end{aligned}$$

and, for every object Y of $\mathcal{C}$, the functor

$$\begin{aligned} \mathcal{C} &\to \mathcal{C} \\ X &\mapsto X \otimes Y \end{aligned}$$

admit right adjoints. In this case, we get pairs of adjoint functors

$$\begin{aligned} \mathcal{C} &\to \mathcal{C} & \qquad \mathcal{C} &\to \mathcal{C} \\ Y &\mapsto X \otimes Y & \qquad Z &\mapsto \underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{C}}(X, Z) \end{aligned}$$

and

$$\begin{aligned} \mathcal{C} &\to \mathcal{C} & \qquad \mathcal{C} &\to \mathcal{C} \\ X &\mapsto X \otimes Y & \qquad Z &\mapsto \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{C}}(Y, Z). \end{aligned}$$

Moreover, $\underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{C}}$ and $\underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{C}}$ extend to functors

$$\underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{C}}, \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

and, if X, Y and Z are three objects, we get natural bijections

$$\mathsf{Hom}_{\mathcal{C}}(X, \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{C}}(Y, Z)) \simeq \mathsf{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \mathsf{Hom}_{\mathcal{C}}(Y, \underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{C}}(X, Z)).$$

Remark A.7. — Let $\mathcal{C}$ be a monoidal category. If $\mathcal{C}$ is biclosed, then its tensor product preserves colimits in each variable. By a classical adjoint theorem, the converse holds provided that the category $\mathcal{C}$ is locally presentable.

Proposition A.8. — *Let $\mathcal{M}$ be a model category endowed with a biclosed monoidal category structure. Then the following conditions are equivalent:*

i) the tensor product satisfies the pushout-product axiom, *ii) for every cofibration $i : A \to B$ and every fibration $p : X \to Y$, the induced map*

$$\underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{M}}(B, X) \to \underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{M}}(A, X) \times_{\underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{M}}(A,Y)} \underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{M}}(B, Y)$$

is a fibration that is trivial if either i or p is,

iii) for every cofibration $j : C \to D$ and every fibration $p : X \to Y$, the induced map

$$\underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{M}}(D, X) \to \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{M}}(C, X) \times_{\underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{M}}(C,Y)} \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{M}}(D, Y)$$

is a fibration that is trivial if either j or p is.

Proof. — See for instance [**10**, Lemma 4.2.2]. □

A.9. — A *biclosed monoidal model category* is a monoidal model category whose underlying monoidal category is biclosed. Our example of interest in this paper is the folk model category structure on ω-$\mathcal{C}at$ (or more generally (m, n)-$\mathcal{C}at$) endowed with the Gray tensor product (see Theorems 5.6 and 2.10). Since tensoring any object by the initial object gives the initial object in a biclosed monoidal model category, the hypothesis of Proposition A.5 are satisfied in such a model category and the monoidal tensor $\otimes$ thus admits a total right derived functor $\otimes^{\mathbb{L}}$.

Proposition A.10. — *Let $\mathcal{M}$ be a biclosed monoidal model category. Then, if X is a cofibrant object of $\mathcal{M}$, the adjoint pair*

$$\begin{aligned} \mathcal{M} &\to \mathcal{M} \\ Y &\mapsto X \otimes Y \end{aligned} \qquad\qquad \begin{aligned} \mathcal{M} &\to \mathcal{M} \\ Z &\mapsto \underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{M}}(X, Z) \end{aligned}$$

is a Quillen pair and, likewise, if Y is a cofibrant object of $\mathcal{M}$, the adjoint pair

$$\begin{aligned} \mathcal{M} &\to \mathcal{M} \\ X &\mapsto X \otimes Y \end{aligned} \qquad\qquad \begin{aligned} \mathcal{M} &\to \mathcal{M} \\ Z &\mapsto \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{M}}(Y, Z) \end{aligned}$$

is a Quillen pair.

Proof. — It suffices to show that the left adjoints respect cofibrations and trivial cofibrations. This follows from the pushout-product axiom (see Remark A.2). □

***Theorem A.11* (Hovey)**. — *Let $\mathcal{M}$ be a biclosed monoidal model category. Then the monoidal category structure on* $\mathrm{Ho}(\mathcal{M})$ *defined by the derived tensor product is biclosed. Moreover, the functors*

$$\underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{M}}, \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{M}} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$$

admit total right derived functors and we have

$$\mathbb{R}\underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathcal{M}} = \underline{\mathsf{Hom}}^{\mathrm{l}}_{\mathrm{Ho}(\mathcal{M})} \quad \textit{and} \quad \mathbb{R}\underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathcal{M}} = \underline{\mathsf{Hom}}^{\mathrm{r}}_{\mathrm{Ho}(\mathcal{M})}\,.$$

Proof. — This is [**10**, Theorem 4.3.2]. Let us just briefly recall why these functors admit total right derived functors: one deduces from Proposition A.8 that these functors preserve trivial fibrations between fibrant objects; by Ken Brown's lemma, this implies that they preserve weak equivalences between fibrant objects and hence that they admit total right derived functors. □

We now move on to locally biclosed monoidal categories, as introduced in [**2**].

A.12. — Let $\mathcal{C}$ be a monoidal category. Let us denote by $\star$ the tensor product of $\mathcal{C}$ and suppose that its tensor unit is an initial object $\varnothing$. If X and Y are two objects, we get morphisms
$$X \xrightarrow{\iota_1} X \star Y \xleftarrow{\iota_2} Y$$
by precomposing the morphisms
$$X \star \varnothing \xrightarrow{X \star \varnothing_Y} X \star Y \xleftarrow{\varnothing_X \star Y} \varnothing \star Y,$$
where $\varnothing_Z$ denotes the unique morphism from $\varnothing$ to Z, with the unit constraints. Using these morphisms, we obtain, for every object X of $\mathcal{C}$, a functor
$$\begin{aligned} \mathcal{C} &\to {}_{X\backslash}\mathcal{C} \\ Y &\mapsto (X \star Y, \iota_1) \end{aligned}$$
and, for every object Y of $\mathcal{C}$, a functor
$$\begin{aligned} \mathcal{C} &\to {}_{Y\backslash}\mathcal{C} \\ X &\mapsto (X \star Y, \iota_2). \end{aligned}$$

We say that $\mathcal{C}$ is *locally biclosed* if its tensor unit is an initial object and if the two above functors admit right adjoints. In this case, we thus get pairs of adjoint functors
$$\begin{aligned} \mathcal{C} &\to {}_{X\backslash}\mathcal{C} \\ Y &\mapsto (X \star Y, \iota_1) \end{aligned} \qquad\qquad \begin{aligned} {}_{X\backslash}\mathcal{C} &\to \mathcal{C} \\ (Z, X \xrightarrow{u} Z) &\mapsto {}_{u\backslash}Z \end{aligned}$$
and
$$\begin{aligned} \mathcal{C} &\to {}_{Y\backslash}\mathcal{C} \\ X &\mapsto (X \star Y, \iota_2) \end{aligned} \qquad\qquad \begin{aligned} {}_{Y\backslash}\mathcal{C} &\to \mathcal{C} \\ (Z, Y \xrightarrow{v} Z) &\mapsto Z_{/v}. \end{aligned}$$
By abuse of notation, we will often denote ${}_{u\backslash}Z$ by ${}_{X\backslash}Z$ and, similarly, $Z_{/v}$ by $Z_{/Y}$. These functors are called the *slice functors*. By definition, we have natural bijections
$$\mathsf{Hom}_{{}_{X\backslash}\mathcal{C}}((X \star Y, \iota_1), (Z, u)) \simeq \mathsf{Hom}_{\mathcal{C}}(Y, {}_{X\backslash}Z)$$
and
$$\mathsf{Hom}_{{}_{Y\backslash}\mathcal{C}}((X \star Y, \iota_2), (Z, v)) \simeq \mathsf{Hom}_{\mathcal{C}}(X, Z_{/Y}).$$
Similarly to what happened in the case of biclosed monoidal categories, the functors $(Z, u : X \to Z) \mapsto {}_{X\backslash}Z$ and $(Z, v : Y \to Z) \mapsto Z_{/Y}$ can be made functorial in X and Y, respectively. More precisely, they canonically extend to functors
$$\begin{aligned} \mathrm{Tw}(\mathcal{C}) &\to \mathcal{C} \\ X \to Z &\mapsto {}_{X\backslash}Z \end{aligned} \qquad\qquad \begin{aligned} \mathrm{Tw}(\mathcal{C}) &\to \mathcal{C} \\ Y \to Z &\mapsto Z_{/Y}, \end{aligned}$$
where $\mathrm{Tw}(\mathcal{C})$ denotes the twisted arrow category of $\mathcal{C}$. (Recall that the objects of $\mathrm{Tw}(\mathcal{C})$ are arrows $f : X \to Y$ of $\mathcal{C}$ and that a morphism of $\mathrm{Tw}(\mathcal{C})$ from an object

$f : X \to Y$ to an object $f' : X' \to Y'$ is a pair of morphisms $g : X' \to X$ and $h : Y \to Y'$ of $\mathcal{C}$ making the square

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "g"'] \arrow[d, "f'"] \\ Y \arrow[r, "h"'] & Y' \end{tikzcd}$$

commute.) Indeed, if

$$\begin{tikzcd} X \arrow[d, "u"'] & X' \arrow[l, "g"'] \arrow[d, "u'"] \\ Z \arrow[r, "h"'] & Z' \end{tikzcd}$$

is a commutative square, one defines a morphism

$$(g^*, h_*) : {}_{X}\backslash Z \to {}_{X'}\backslash Z'$$

using the Yoneda lemma. If T is any object of $\mathcal{C}$, then

$$\begin{aligned} \mathsf{Hom}_{\mathcal{C}}(T, {}_{X}\backslash Z) &\simeq \mathsf{Hom}_{X\backslash\mathcal{C}}((X \star T, \iota_1), (Z, u)) \\ \mathsf{Hom}_{\mathcal{C}}(T, {}_{X'}\backslash Z') &\simeq \mathsf{Hom}_{X'\backslash\mathcal{C}}((X' \star T, \iota_1), (Z', u')) \end{aligned}$$

and the natural map

$$\mathsf{Hom}_{\mathcal{C}}(g \star T, h) : \mathsf{Hom}_{\mathcal{C}}(X \star T, Z) \to \mathsf{Hom}_{\mathcal{C}}(X' \star T, Z')$$

induces the desired morphism. Note that this morphism (g^*, h_*) is the diagonal of the commutative square

$$\begin{tikzcd} {}_{X}\backslash Z \arrow[r, "h_*"] \arrow[d, "g^*"'] & {}_{X}\backslash Z' \arrow[d, "g^*"] \\ {}_{X'}\backslash Z \arrow[r, "h_*"'] & {}_{X'}\backslash Z' \end{tikzcd} \quad ,$$

where $g^* = (g^*, 1_*)$ and $h_* = (1^*, h_*)$. A similar construction applies to the other slice functor.

Remark A.13. — Let $\mathcal{C}$ be a monoidal category. If $\mathcal{C}$ is locally biclosed, then its tensor product preserves connected colimits in each variable (as the forgetful functor $\mathcal{C} \to {}_{Z}\backslash\mathcal{C}$ preserves these colimits). By a classical adjoint theorem, the converse holds provided that the category $\mathcal{C}$ is locally presentable (as colimits in ${}_{Z}\backslash\mathcal{C}$ can be computed as connected colimits in $\mathcal{C}$).

A.14. — A *locally biclosed monoidal model category* is a monoidal model category whose underlying monoidal category is locally biclosed. Our example of interest in this paper is the folk model category structure on ω-$\mathcal{C}at$ (or more generally n-$\mathcal{C}at$) endowed with the join (see Theorems 7.14 and 7.2). Note that in the locally biclosed setting, the unit axiom is a consequence of the pushout-product axiom. This follows from Remark A.2 as the tensor unit is the initial object and is thus cofibrant. Moreover, Proposition A.5 shows that the monoidal tensor $\star$ of such a model category admits a total right derived functor $\star^{\mathbb{L}}$.

Proposition A.15. — *Let $\mathcal{M}$ be a locally biclosed monoidal model category. For every cofibrant object X of $\mathcal{M}$, the adjoint pair*

$$\begin{aligned} \mathcal{M} &\to {}_{X}\backslash\mathcal{M} & \qquad {}_{X}\backslash\mathcal{M} &\to \mathcal{M} \\ Y &\mapsto (X \star Y, \iota_1) & (Z, X \to Z) &\mapsto {}_{X}\backslash Z \end{aligned}$$

is a Quillen pair and, likewise, for every cofibrant object Y of $\mathcal{M}$, the adjoint pair

$$\begin{aligned} \mathcal{M} &\to {}_{Y}\backslash\mathcal{M} & \qquad {}_{Y}\backslash\mathcal{M} &\to \mathcal{M} \\ X &\mapsto (X \star Y, \iota_2) & (Z, Y \to Z) &\mapsto Z/_{Y} \end{aligned}$$

is a Quillen pair.

Proof. — It suffices to show that the left adjoints respect cofibrations and trivial cofibrations. As the cofibrations and trivial cofibrations of ${}_{Z}\backslash\mathcal{M}$ are defined using the forgetful functor ${}_{Z}\backslash\mathcal{M} \to \mathcal{M}$, this follows from the pushout-product axiom by Remark A.2. □

The goal of the rest of this appendix is to derive the slice functors of a locally biclosed monoidal model category $\mathcal{M}$, as functors of source $\mathrm{Tw}(\mathcal{M})$, where the weak equivalences of $\mathrm{Tw}(\mathcal{M})$ are the level-wise weak equivalences. Unfortunately, the category $\mathrm{Tw}(\mathcal{M})$ is neither finitely cocomplete (it does not even have an initial object) nor finitely complete in general, and therefore cannot be endowed with a model category structure. We will see that it can be endowed with a right simplicial derivability structure in the sense of Kahn and Maltsiniotis (see [**12**, Definition 6.7]) and that this is enough to derive the slice functors.

We start by recalling this notion of right simplicial derivability structure and the corresponding result of derivation.

A.16. — A *localizer*, or *relative category*, is a pair $(\mathcal{C}, \mathcal{W})$, where $\mathcal{C}$ is a category and $\mathcal{W}$ is a class of morphisms of $\mathcal{C}$ called *weak equivalences*. Such a localizer is said to be *multiplicative* if $\mathcal{W}$ contains all the identities and is stable under composition. In this case, the class $\mathcal{W}$ can be identified with a subcategory of $\mathcal{C}$ with same objects as $\mathcal{C}$.

A *morphism* from a localizer $(\mathcal{C}, \mathcal{W})$ to a localizer $(\mathcal{C}', \mathcal{W}')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ such that $F(\mathcal{W}) \subset \mathcal{W}'$.

If $(\mathcal{C}, \mathcal{W})$ is a localizer and I is a small category, then we get a localizer $(\mathcal{C}_I, \mathcal{W}_I)$, where $\mathcal{C}_I$ denotes the category $\underline{\mathsf{Hom}}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$ and $\mathcal{W}_I$ the class of natural transformations between these functors which are object-wise weak equivalences. This construction is functorial in I in an obvious way: if $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ is a morphism of localizers, we get a morphism $F_I : (\mathcal{C}_I, \mathcal{W}_I) \to (C'_I, \mathcal{W}'_I)$.

A.17. — Fix $K : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}, \mathcal{W})$ a morphism of multiplicative localizers and denote by $K^\flat$ the induced functor $K^\flat : \mathcal{W}_0 \to \mathcal{W}$. If X is an object of $\mathcal{C}$, the *category of right K-resolutions* of X is the comma category $X \downarrow K^\flat$, that is, the category whose objects are pairs $(Y, X \xrightarrow{w} KY)$, where Y is an object of $\mathcal{C}_0$ and w is a weak equivalence of $\mathcal{C}$, and whose morphisms from an object (Y, w) to an object (Y', w') are the weak equivalences $w_0 : Y \to Y'$ of $\mathcal{C}_0$ such that $K(w_0)w = w'$.

If I is a small category and $F : I \to \mathcal{C}$ is a functor, then, by considering the induced morphism of localizers K_I, we get a notion of category of right K_I-resolutions for F. In particular, considering $I = \{0 < 1\}$, we get a notion of *category of right K-resolutions of an arrow* of $\mathcal{C}$, and taking $I = \{0 < 1 < 2\}$, we get a notion of *category of right K-resolutions of a pair of composable arrows* of $\mathcal{C}$.

A.18. — Let $(\mathcal{C}, \mathcal{W})$ be a multiplicative localizer. A *right simplicial derivability structure* on $(\mathcal{C}, \mathcal{W})$ consists of a multiplicative localizer $(\mathcal{C}_0, \mathcal{W}_0)$ and a morphism of localizers $K : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}, \mathcal{W})$ satisfying the following conditions:

(a) for every object X of $\mathcal{C}$, the category of right K-resolutions of X is 1-connected (that is, simply connected and non-empty),

(b) for every arrow f of $\mathcal{C}$, the category of right K-resolutions of f is 0-connected (that is, connected and non-empty),

(c) for every pair (g, f) of composable arrows of $\mathcal{C}$, the category of right K-resolutions of (g, f) is -1-connected (that is, non-empty).

Example A.19. — If $\mathcal{M}$ is a model category, then $(\mathcal{M}, \mathcal{W})$, where $\mathcal{W}$ is the class of weak equivalences of $\mathcal{M}$, is naturally endowed with a right simplicial derivability structure $K : (\mathcal{M}_0, \mathcal{W}_0) \to (\mathcal{M}, \mathcal{W})$, where $\mathcal{M}_0$ denotes the full subcategory of $\mathcal{M}$ consisting of fibrant objects and $\mathcal{W}_0$ the class of weak equivalences between fibrant objects (see the "table of implications" at the very end of [**12**]).

***Proposition A.20* (Kahn–Maltsiniotis)**. — *Let $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ be a morphism of localizers. If there exists a right simplicial derivability structure $K : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}, \mathcal{W})$ on $(\mathcal{C}, \mathcal{W})$ such that $FK(\mathcal{W}_0) \subset \mathcal{W}'$, then F admits a total right derived functor $\mathbb{R}F : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}'[\mathcal{W}'^{-1}]$.*

Proof. — See [**12**, Proposition 5.9 and paragraph 6.8]. □

We will now prove a general result allowing to lift a right simplicial derivability structure along a discrete opfibration, result that we will then apply to the discrete opfibration $\mathrm{Tw}(\mathcal{M}) \to \mathcal{M}^{\mathrm{op}} \times \mathcal{M}$.

Proposition A.21. — *Let $(\mathcal{C}, \mathcal{W})$ be a multiplicative localizer endowed with a right simplicial derivability structure $K : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}, \mathcal{W})$ and let $p : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a discrete opfibration. Set $\widetilde{\mathcal{W}} = p^{-1}(\mathcal{W})$. Then $(\widetilde{\mathcal{C}}, \widetilde{\mathcal{W}})$ is endowed with a natural simplicial derivability structure $\widetilde{K} : (\widetilde{\mathcal{C}}_0, \widetilde{\mathcal{W}}_0) \to (\widetilde{\mathcal{C}}, \widetilde{\mathcal{W}})$, obtained by pulling back K along p.*

Proof. — Let I be a small category and let $\widetilde{F} : I \to \widetilde{\mathcal{C}}$ be a functor. We are going to show that the categories of right $\widetilde{K}_I$-resolutions of $\widetilde{F}$ and of right K_I-resolutions of $p\widetilde{F}$ are isomorphic. This will immediately imply the result.

By definition, we have a pullback square

$$\begin{tikzcd} (\widetilde{\mathcal{C}}_0, \widetilde{\mathcal{W}}_0) \arrow[r, "\widetilde{K}"] \arrow[d] & (\widetilde{\mathcal{C}}, \widetilde{\mathcal{W}}) \arrow[d, "p"] \\ (\mathcal{C}_0, \mathcal{W}_0) \arrow[r, "K"'] & (\mathcal{C}, \mathcal{W}) \end{tikzcd}$$

in the category of localizers. Note that pullbacks in this category are computed component-wise. By applying the $\underline{\mathsf{Hom}}(I, -)$ functor, we get a commutative square

$$\begin{tikzcd} ((\widetilde{\mathcal{C}}_0)_I, (\widetilde{\mathcal{W}}_0)_I) \arrow[r, "\widetilde{K}_I"] \arrow[d] & (\widetilde{\mathcal{C}}_I, \widetilde{\mathcal{W}}_I) \arrow[d, "p_I"] \\ ((\mathcal{C}_0)_I, (\mathcal{W}_0)_I) \arrow[r, "K_I"'] & (\mathcal{C}_I, \mathcal{W}_I) \end{tikzcd} \quad ,$$

that is easily seen to still be a pullback square. As the $\underline{\mathsf{Hom}}(I, -)$ functor preserves discrete opfibrations, the functor $p_I : \widetilde{\mathcal{C}}_I \to \mathcal{C}_I$ and therefore its restriction $(p_I)^\flat : \widetilde{\mathcal{W}}_I \to \mathcal{W}_I$ are still discrete opfibrations.

By definition, the category of right $\widetilde{K}_I$-resolutions of $\widetilde{F} : I \to \widetilde{\mathcal{C}}$ is the comma category $\widetilde{F} \downarrow (\widetilde{K}_I)^\flat$, while the category of right K_I-resolutions of $p_I(\widetilde{F}) = p\widetilde{F} : I \to \mathcal{C}$ is the comma category $p_I(\widetilde{F}) \downarrow (K_I)^\flat$. The result thus follows from the following lemma, applied to the pullback square

$$\begin{tikzcd} (\widetilde{\mathcal{W}}_0)_I \arrow[r, "(\widetilde{K}_I)^\flat"] \arrow[d] & \widetilde{\mathcal{W}}_I \arrow[d, "(p_I)^\flat"] \\ (\mathcal{W}_0)_I \arrow[r, "(K_I)^\flat"'] & \mathcal{W}_I \end{tikzcd} \quad ,$$

lemma which is probably well known and whose proof is left as an easy exercise to the reader. $\square$

Lemma A.22. — *Let*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "G"] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\ \mathcal{B}' \arrow[r, "F"'] & \mathcal{B} \end{tikzcd}$$

be a pullback square of categories, where p is a discrete opfibration. Then, for every object X of $\mathcal{X}$, the functor p induces an isomorphism between the comma categories $X \downarrow G$ and $p(X) \downarrow F$.

A.23. — Let $\mathcal{M}$ be a model category. We will say that a morphism

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d, "f'"] \\ Y \arrow[u, leftarrow, "f"] \arrow[r, "h"'] & Y' \end{tikzcd}$$

of $\mathrm{Tw}(\mathcal{M})$ from f to f' is

- a *weak equivalence* if g and h are,
- a *fibration* if g is a cofibration and h is a fibration.

The category $\mathrm{Tw}(\mathcal{M})$ admits as a terminal object the unique arrow $\varnothing \to *$ from the initial object of $\mathcal{M}$ to the terminal object of $\mathcal{M}$, and we will say that an object $X \to Y$ of $\mathrm{Tw}(\mathcal{M})$ is *fibrant* if the unique morphism from this object to the terminal object is a fibration. This amounts to saying that X is cofibrant and Y is fibrant.

We will denote by $(\mathrm{Tw}(\mathcal{M}), \widetilde{\mathcal{W}})$ the resulting localizer and by $(\mathrm{Tw}(\mathcal{M})_0, \widetilde{\mathcal{W}_0})$ the induced localizer on the full subcategory of $\mathrm{Tw}(\mathcal{M})$ consisting of fibrant objects.

Proposition A.24. — *If $\mathcal{M}$ is a model category, then the inclusion morphism*

$$(\mathrm{Tw}(\mathcal{M})_0, \widetilde{\mathcal{W}_0}) \hookrightarrow (\mathrm{Tw}(\mathcal{M}), \widetilde{\mathcal{W}})$$

is a right simplicial derivability structure.

Proof. — It is immediate that the functor

$$\begin{aligned} \mathrm{Tw}(\mathcal{M}) &\to \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \\ X \to Y &\mapsto (X, Y) \end{aligned}$$

is a discrete opfibration. We thus get the result by applying Proposition A.21 to this functor and to the right simplicial derivability structure associated to the model category $\mathcal{M}^{\mathrm{op}} \times \mathcal{M}$ (see Example A.19). □

To use Proposition A.20 to derive the slice functors, we now need to prove that these functors preserve weak equivalences between fibrant objects. To do so, we will generalize Proposition A.8 to the locally biclosed setting.

A.25. — Let $\mathcal{C}$ be a locally biclosed monoidal category. If $i : A \to B$ and $j : C \to D$ are two morphisms of $\mathcal{C}$, note that the morphism

$$i \star' j : B \star C \amalg_{A \star C} A \star D \to B \star D$$

is naturally above both B and D. If now $p : (X, f) \to (Y, g)$ is a morphisms of ${}_{B}\backslash\mathcal{C}$, using i and p we get a morphism

$$i \backslash' p : {}_{B}\backslash^{X} \to {}_{A}\backslash^{X} \times_{{}_{A}\backslash^{Y}} {}_{B}\backslash^{Y}$$

induced by the commutative square

$$\begin{tikzcd} B\backslash X \arrow[r, "p_*"] \arrow[d, "i^*"'] & B\backslash Y \arrow[d, "i^*"] \\ A\backslash X \arrow[r, "p_*"'] & A\backslash Y \end{tikzcd} .$$

Similarly, from j and a morphism $p : (X, f) \to (Y, g)$ of $D\backslash\mathcal{C}$, we get a morphism

$$p/'j : X/D \to X/C \times_{Y/C} Y/D.$$

***Lemma A.26* (Joyal).** — *Let $\mathcal{C}$ be a locally biclosed monoidal category. If $i : A \to B$ and $j : C \to D$ are two morphisms of $\mathcal{C}$ and $p : X \to Y$ is a morphism of $\mathcal{C}$ above D, then we have*

$$i \star' j \perp_{D\backslash\mathcal{C}} p \quad \textit{if and only if} \quad j \perp_{\mathcal{C}} i\backslash' p \quad \textit{if and only if} \quad i \perp_{\mathcal{C}} p/'j,$$

where $\perp_{\mathcal{D}}$ denotes the relation of weak orthogonality in the category $\mathcal{D}$.

Proof. — The lemma is inspired by [**11**, Lemma 3.6], whose proof applies *mutatis mutandis.* □

***Proposition A.27*.** — *Let $\mathcal{M}$ be a model category endowed with a locally biclosed monoidal category structure. Then the following conditions are equivalent:*

i) the tensor product $\star$ satisfies the pushout-product axiom,

ii) for every cofibration $i : A \to B$, every fibration $p : X \to Y$ and every map $f : B \to X$, the induced map

$$i\backslash' p : B\backslash X \to A\backslash X \times_{A\backslash Y} B\backslash Y$$

is a fibration that is trivial if either i or p is,

iii) for every cofibration $j : C \to D$, every fibration $p : X \to Y$ and every map $f : D \to X$, the induced map

$$p/'j : X/D \to X/C \times_{Y/C} Y/D$$

is a fibration that is trivial if either j or p is.

Proof. — This follows directly from the previous lemma and the fact that

$$\begin{aligned} i \star' j \perp_{\mathcal{M}} p \quad &\text{if and only if} \quad \text{for every } f : B \to X, \text{ we have } i \star' j \perp_{B\backslash\mathcal{M}} p, \\ &\text{if and only if} \quad \text{for every } f : D \to X, \text{ we have } i \star' j \perp_{D\backslash\mathcal{M}} p. \end{aligned}$$ □

***Proposition A.28*.** — *If $\mathcal{M}$ is a locally biclosed monoidal model category, then the functors*

$$\begin{aligned} \mathrm{Tw}(\mathcal{M}) &\to \mathcal{M} & \qquad \mathrm{Tw}(\mathcal{M}) &\to \mathcal{M} \\ X \to Z &\mapsto X\backslash Z & \qquad Y \to Z &\mapsto Z/Y \end{aligned}$$

both send fibrations (resp. *trivial fibrations*) *between fibrant objects to fibrations* (resp. *trivial fibrations*)*. Moreover, they preserve weak equivalences between fibrant objects.*

Proof. — Let us prove the result for the first functor, the proof for the second one being similar. Let

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d, "u'"] \\ Z \arrow[u, leftarrow, "u"] \arrow[r, "h"'] & Z' \end{tikzcd}$$

be a morphism of $\mathrm{Tw}(\mathcal{M})$ between fibrant objects u and u'. The morphism

$$(g^*, h_*) : X\backslash Z \to X'\backslash Z'$$

factors as

$$X\backslash Z \xrightarrow{g^*} X'\backslash Z \xrightarrow{h_*} X'\backslash Z'.$$

These morphisms g^* and h_* are the images of g and h by the functors

$$\begin{aligned} (\mathcal{M}/Z)^{\mathrm{op}} &\to \mathcal{M} \\ (X, X \to Z) &\mapsto X\backslash Z \end{aligned} \qquad\qquad \begin{aligned} X'\backslash\mathcal{M} &\to \mathcal{M} \\ (Z, X' \to Z) &\mapsto X'\backslash Z, \end{aligned}$$

and it therefore suffices to show that the first of these functors sends cofibrations (resp. trivial cofibrations) of $\mathcal{M}/Z$ to fibrations (resp. trivial fibrations) of $\mathcal{M}$ and that the second one preserves fibrations and trivial fibrations. Note that by Ken Brown's lemma (which cannot be applied directly to $\mathrm{Tw}(\mathcal{M})$), this will imply that the first functor preserves weak equivalences between cofibrant objects of $\mathcal{M}/Z$ (an object of $\mathcal{M}/Z$ being cofibrant if its underlying object in $\mathcal{M}$ is) and that the second functor preserves weak equivalences between fibrant objects in $X'\backslash\mathcal{M}$ (an object of $X'\backslash\mathcal{M}$ being fibrant if its underlying object in $\mathcal{M}$ is), thereby proving the second assertion.

For the first functor, observe that the morphism g^* can be identified with the morphism

$$g\backslash' p : X\backslash Z \to X'\backslash Z \times_{X'\backslash *} X\backslash *$$

of paragraph A.25, where $p : Z \to *$ denotes the unique morphism from Z to the terminal object $*$. As Z is fibrant, Proposition A.27 implies that this first functor sends cofibrations (resp. trivial cofibrations) of $\mathcal{M}/Z$ to fibrations (resp. trivial fibrations) of $\mathcal{M}$. As for the second functor, since X' is cofibrant, it preserves fibrations and trivial fibrations by Proposition A.15. □

Theorem A.29. — *If $\mathcal{M}$ is a locally biclosed monoidal model category, then the functors*

$$\begin{aligned} \mathrm{Tw}(\mathcal{M}) &\to \mathcal{M} \\ X \to Z &\mapsto X\backslash Z \end{aligned} \qquad\qquad \begin{aligned} \mathrm{Tw}(\mathcal{M}) &\to \mathcal{M} \\ Y \to Z &\mapsto Z/Y \end{aligned}$$

both admit total right derived functors.

Proof. — Since by the previous proposition these functors preserve weak equivalences between fibrant objects, the result follows from the derivability condition of Kahn and Maltsiniotis (Proposition A.20) applied to the right simplicial derivability structure of Proposition A.24. ☐

Corollary A.30. — *The functors*

$$\begin{aligned} \operatorname{Tw}(\omega\text{-}\mathcal{C}at) &\to \omega\text{-}\mathcal{C}at & \qquad \operatorname{Tw}(\omega\text{-}\mathcal{C}at) &\to \omega\text{-}\mathcal{C}at \\ X \xrightarrow{u} Z &\mapsto u\backslash Z & Y \xrightarrow{v} Z &\mapsto Z \overset{\mathrm{co}}{/} v \end{aligned}$$

(see paragraph 7.3 for the notation), where ω-$\mathcal{C}at$ is endowed with the folk model category structure, admit total right derived functors.

Proof. — This follows from the previous theorem applied to the folk model category structure on ω-$\mathcal{C}at$ endowed with the join (see Theorems 7.14 and 7.2). ☐

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