LAX FUNCTORIALITIES OF THE COMMA CONSTRUCTION FOR ω-CATEGORIES

by

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Abstract. — Motivated by the Grothendieck construction, we study the functorialities of the comma construction for strict ω-categories. To state the most general functorialities, we use the language of Gray ω-categories, that is, categories enriched in the category of strict ω-categories endowed with the oplax Gray tensor product. Our main result is that the comma construction of strict ω-categories defines a Gray ω-functor, that is, a morphism of Gray ω-categories. To makes sense of this statement, we prove that slices of Gray ω-categories exist. Coming back to the Grothendieck construction, we propose a definition in terms of the comma construction and, as a consequence, we get that the Grothendieck construction of strict ω-categories defines a Gray ω-functor. Finally, as a by-product, we get a notion of Grothendieck construction for Gray ω-functors, which we plan to investigate in future work.

Introduction

The starting point: the Grothendieck construction

The classical Grothendieck construction defines, for every small category I, a functor

$$\textstyle\int_I \colon \underline{\mathsf{Hom}}(I^{\mathrm{o}}, \mathcal{C}at) \to \mathcal{C}at$$

that sends a functor $F\colon I^{\mathrm{o}} \to \mathcal{C}at$, where I^{o} denotes the opposite category of I and $\mathcal{C}at$ the category of small categories, to the so-called *Grothendieck construction* $\int_I F$ of F. Here $\underline{\mathsf{Hom}}$ denotes the cartesian internal hom of $\mathcal{C}at$, whose morphisms are strict natural transformations. But the functorialities of the Grothendieck construction are more general. First, if $F, G\colon I^{\mathrm{o}} \to \mathcal{C}at$ are two functors of this kind and $\alpha\colon F \Rightarrow G$ is an *oplax* transformation (that is, roughly speaking, a transformation where the naturality squares only commute up to a non-invertible 2-cell), then one can still

2020 ***Mathematics Subject Classification.*** — 18A25, 18D20, 18N20, 18N30.
Keywords. — ω-categories, comma ω-categories, Gray ω-categories, Grothendieck construction, slice Gray ω-categories, strict ω-categories.

integrate α to obtain a functor $\int_I \alpha\colon \int_I F \to \int_I G$. Second, the construction is also functorial in I. Combining these, we get a functoriality

$$\begin{array}{ccc} I^{\mathrm{o}} & \xrightarrow{u^{\mathrm{o}}} & J^{\mathrm{o}} \\ & {\scriptstyle\alpha}\Rightarrow & \\ {\scriptstyle F}\searrow & & \swarrow{\scriptstyle G} \\ & \mathcal{C}at & \end{array} \qquad \mapsto \qquad \textstyle\int_I F \xrightarrow{\int(u,\alpha)} \int_J G \quad ,$$

where α is an oplax transformation.

The purpose of this paper is to study higher generalizations of these functorialities in the setting of strict ω-categories. Our original motivation was to investigate the homotopical properties of the Grothendieck construction for strict ω-categories, and particularly the generalization of a theorem by Thomason [**9**], which will be addressed in a separate paper [**2**].

From the Grothendieck construction to comma ω-categories

Let ω-$\mathcal{C}at$ denote the category of strict ω-categories and strict ω-functors. Using its cartesian internal hom, this category can be promoted to a strict ω-category ω-$\mathcal{C}at_{\mathrm{cart}}$, whose 2-cells are strict transformations and whose higher cells are strict higher transformations. If now $F\colon I^{\mathrm{o}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is a strict ω-functor, where I is a strict ω-category and I^{o} denotes the dual ω-category obtained by reversing the orientation of all the cells of I, then a Grothendieck construction $\int_I F$ was defined by Warren in his work on the model of strict ω-groupoids for dependent type theory [**10**].

However, the definition given by Warren is unsatisfactory because it relies on explicit and complicated formulas. We propose to *define* the Grothendieck construction of $F\colon I^{\mathrm{o}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ as the universal ω-category $\int_I F$ endowed with a 2-square

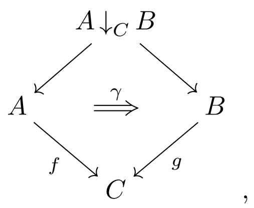
$$\begin{array}{ccc} & (\int_I F)^{\mathrm{o}} & \\ \swarrow & & \searrow \\ \mathrm{D}_0 & \overset{\gamma}{\Longrightarrow} & I^{\mathrm{o}} \\ {\scriptstyle c_{\mathrm{D}_0}}\searrow & & \swarrow{\scriptstyle F} \\ & \omega\text{-}\mathcal{C}at_{\mathrm{cart}} & \end{array} \quad ,$$

where D_0 denotes the terminal ω-category, c_{D_0} the constant ω-functor of value D_0 and γ an oplax transformation. This type of universal 2-squares was already studied by the first-named author and Maltsiniotis [**3**] and is a straightforward generalization of the classical comma construction of two functors $f\colon A \to C$ and $g\colon B \to C$, usually denoted by $f \downarrow g$. More precisely, we have

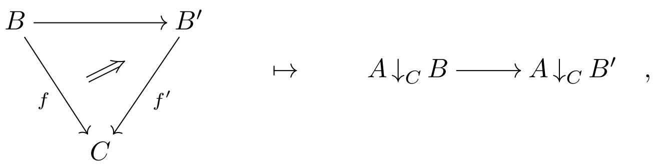
$$\textstyle\int_I F = (c_{\mathrm{D}_0} \downarrow F)^{\mathrm{o}} \,,$$

where $\downarrow$ denotes the *oplax* comma construction. Although these definitions are abstract, explicit formulas can be extracted, and one can recover Warren's formulas from this abstract point of view.

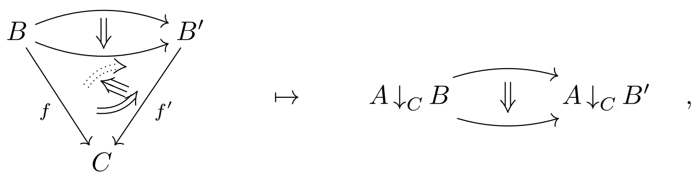
We are thus led to consider the following more general case. Let A, B and C be three strict ω-categories, and let $f\colon A \to C$ and $g\colon B \to C$ be two strict ω-functors. We can form the (oplax) comma ω-category $A\downarrow_C B$, that is, the universal strict ω-category equipped with a 2-square



where γ is an oplax transformation. Our goal is now to study the functorialities of $A\downarrow_C B$ in $g\colon B \to C$, with $f\colon A \to C$ fixed, and symmetrically, of which the functorialities of the Grothendieck construction are a particular case. The universal property of the comma construction immediately gives a functoriality



where the 2-cell represents an oplax transformation. Working a bit harder, the first-named author and Maltsiniotis got a functoriality



where the 2-cells represent oplax transformations and the 3-cell represents an oplax 2-transformation (also known as an oplax modification), and proved that the comma construction defines a sesquifunctor (see [**3**, Theorem B.2.6]). And now comes the question: what is the general functoriality statement?

The need for Gray ω-categories and their slices

The answer to this question uses the language of Gray ω-categories, which the first-named author introduced with Maltsiniotis in their work on the join construction and the slices [**4**]. Indeed, the diagrams above involving 0-cells, 1-cells, 2-cells and 3-cells actually live in $\omega\text{-}\mathsf{Cat}_{\text{oplax}}$, in which 0-cells are strict ω-categories, 1-cells are strict

ω-functors, 2-cells are oplax transformations, 3-cells are oplax 2-transformations, and so on. But ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ is not an ω-category, *not even a weak one*! Indeed, if

$$\begin{tikzcd} A \ar[r, bend left] \ar[r, bend right] \ar[r, phantom, "\alpha\Downarrow"] & B \ar[r, bend left] \ar[r, bend right] \ar[r, phantom, "\beta\Downarrow"] & C \end{tikzcd}$$

are two oplax transformations, then there are a priori two ways of composing them:

$$(t(\beta) *_0 \alpha) *_1 (\beta *_0 s(\alpha)) \quad \text{and} \quad (\beta *_0 t(\alpha)) *_1 (s(\beta) *_0 \alpha)\,,$$

where s and t denote the source and the target. In general, these two oplax transformations are different! In other words, ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ does not satisfy the exchange rule. What is true is that there is a *non-invertible* canonical oplax 2-transformation

$$(\beta *_0 t(\alpha)) *_1 (s(\beta) *_0 \alpha) \Longrightarrow (t(\beta) *_0 \alpha) *_1 (\beta *_0 s(\alpha))\,,$$

which can be pictured as

$$\left\{\begin{tikzcd}[row sep=small] A \ar[r, bend left] \ar[r, bend right] \ar[r, phantom, "\Downarrow\alpha"] & B \ar[r] & C \\ & *_1 & \\ A \ar[r] & B \ar[r, bend left] \ar[r, bend right] \ar[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}\right\} \Rrightarrow \left\{\begin{tikzcd}[row sep=small] A \ar[r] & B \ar[r, bend left] \ar[r, bend right] \ar[r, phantom, "\Downarrow\beta"] & C \\ & *_1 & \\ A \ar[r, bend left] \ar[r, bend right] \ar[r, phantom, "\Downarrow\alpha"] & B \ar[r] & C \end{tikzcd}\right\}\,.$$

This means that ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ is some kind of oplax ω-category. Formally, ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ is what we call a *Gray ω-category*, that is, a category enriched in ω-$\mathcal{C}at$ endowed with the oplax Gray tensor product. Morphisms of Gray ω-categories are called *Gray ω-functors*.

Let us come back to the comma construction. It seems reasonable to expect that if the correspondence

$$A \downarrow_C - \colon (B, B \to C) \mapsto A \downarrow_C B$$

extends to a Gray ω-functor, then its target should be ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. But what would be the source Gray ω-category? Or, in other words, in which Gray ω-category do the triangles and cones drawn earlier are 1-cells and 2-cells? Obviously, in some kind of slice Gray ω-category ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$ of the Gray ω-category ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ above the strict ω-category C (which is an object of ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$). We prove that slice Gray ω-categories exist, in full generality:

Theorem. — *Let $\mathbb{C}$ be a Gray ω-category and let c be an object of $\mathbb{C}$. Then there is a (natural) Gray ω-category $\mathbb{C}/_c$ of objects of $\mathbb{C}$ over c.*

In the case that $\mathbb{C}$ is a *strict* ω-category (which we can consider as a Gray ω-category where the exchange rule is an equality), we recover the usual notion of slices of strict ω-categories (see for instance [**4**, Chapter 9]). Note that the existence of slices of Gray ω-categories was first conjectured in [**4**, Appendix C].

Our Gray-functoriality results

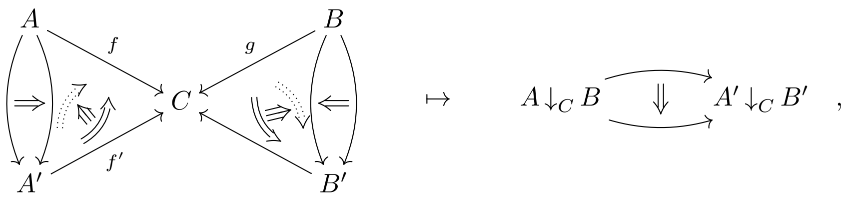
Using slices of Gray ω-categories, we can finally express the desired functoriality of the construction $A \downarrow_C B$, actually both in A and B simultaneously, which is the main result of our paper:

Theorem. — *The oplax comma construction* $- \downarrow_C -$ *defines a Gray ω-functor*

$$- \downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}} \overset{\text{to}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\,,$$

*where the decoration "*to*" indicates a duality by which the slice has to be conjugated.*

This theorem gives the full functorialities of the comma construction. In particular, in low dimensions, we recover the sesquifunctor of [**3**, Theorem B.2.6] and the mapping



where 2-cells represent oplax transformations and the 3-cell represents an oplax 2-transformation. Concretely, our theorem asserts that this mapping generalizes to cones of any dimension and higher oplax transformations, and moreover that these mappings are compatible with all the compositions of Gray slices.

This theorem is an instance of an idea we would like to put forward: *every functorial construction on ω-categories should be promoted to a Gray ω-functor.*

Note that, as the duality appearing in the statement of the theorem shows, dualities of strict and Gray ω-categories play an important and subtle role in this paper, and we study them with great care.

As a direct corollary of the theorem, we get:

Corollary. — *Let A and B be two strict ω-categories. The oplax comma construction restricts to a Gray ω-functor*

$$- \downarrow_C - \colon \underline{\mathsf{Hom}}_{\text{oplax}}(A, C)^{\text{o}} \times \underline{\mathsf{Hom}}_{\text{oplax}}(B, C)^{\text{to}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\,,$$

where $\underline{\mathsf{Hom}}_{\text{oplax}}(X, Y)$ denotes the strict ω-category of strict ω-functors from X to Y, oplax transformations and higher oplax transformations between them.

We also study the functorialities of the oplax comma construction restricted to higher *strict* transformations. The ω-category $\omega\text{-}\mathcal{C}at_{\text{cart}}$ can be seen as a sub-Gray ω-category of $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$ by considering higher strict transformations as particular higher oplax transformations. A priori, we get by restriction a Gray ω-functor

$$- \downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\text{cart}}/C \times \omega\text{-}\mathcal{C}at_{\text{cart}} \overset{\text{to}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\,,$$

but we prove that it actually lands into $\omega\text{-}\mathcal{C}at_{\text{cart}}$:

Proposition. — *The oplax comma construction restricts to a strict ω-functor*

$$-\downarrow_C -\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/} C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Finally, we apply all these results to the particular case of the Grothendieck construction. For that, notice that the ω-category $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is an object of $\omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}}$, the (very large) Gray ω-category of possibly large strict ω-categories, strict ω-functors and lax (higher) transformations between them. We get:

Corollary. — *The Grothendieck construction defines a Gray ω-functor*

$$\begin{aligned}\textstyle\int\colon \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}\{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\\ (F\colon I^{\mathrm{o}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) &\mapsto \textstyle\int_I F\end{aligned}\qquad,$$

$\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}\{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}$ *being the full sub-Gray ω-category of* $\omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}\{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}$ *spanned by those ω-functors $F\colon I^{\mathrm{o}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$, where I is a small ω-category.*

In particular, if we fix a strict ω-category I, the above Gray ω-functor restricts to a Gray ω-functor

$$\textstyle\int_I\colon\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(I^{\mathrm{o}}, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,$$

as well as to a strict ω-functor

$$\textstyle\int_I\colon\ \underline{\mathsf{Hom}}(I^{\mathrm{o}}, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

where $\underline{\mathsf{Hom}}$ is the cartesian internal hom of $\omega\text{-}\mathcal{C}\mathrm{AT}$, the (very large) category of possibly large strict ω-categories.

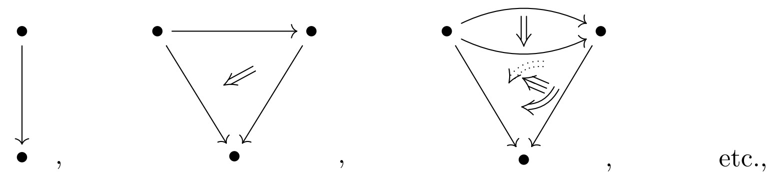
Note that functorialities of the Grothendieck construction in the setting of *weak* ω-categories have been studied recently by Loubaton [**8**]. However, our work is not a particular case of his as, first, Loubaton only deals with the case of a fixed base (weak) ω-category I and, second, he only considers a (weak) sub-ω-category of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(I^{\mathrm{o}}, \omega\text{-}\mathcal{C}at_{\mathrm{cart}})$, containing oplax transformations but not general oplax n-transformations for $n > 1$. In other words, the generalization of our work to weak ω-categories would give more general functorialities than the ones studied by Loubaton(1).

A side remark on cylinders in Gray ω-categories

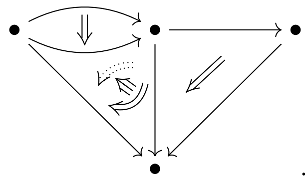
It seems important to bring the reader's attention to the fact that there is a *tour de force* behind the definition of slice Gray ω-category $\mathbb{C}/_C$. Intuitively, to define such

(1) Since we made our work publicly available, preliminary progress in this direction has been made by Gepner and Heine [**5**]

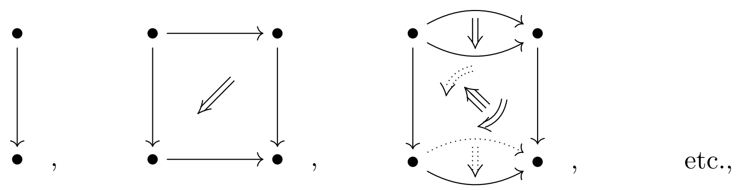
Gray ω-categories, one must make sense in an arbitrary Gray ω-category of pasting diagrams shaped like (higher) cones



as well as defining compositions between those. For example, defining the whiskering operation of a 2-cell with a 1-cell in slice Gray ω-categories amounts to define a "total composite" of the following pasting diagram:



In the case of *strict* ω-categories, the cone-shaped pasting diagrams are considered as degenerate cases of cylinder-shaped pasting diagrams



and the compositions between cone-shaped diagrams are induced by the ones at the level of cylinders. *However*, for Gray ω-categories, as we will explain in an appendix to this paper, it is not possible to define all the cylinder compositions that would be required to then deduce the ones for cones as a particular case. Hence, the fact that we can nonetheless define cone-shaped diagrams and their compositions within a Gray ω-category suggests the existence of an interesting but not yet fully understood theory of pasting diagrams in Gray ω-categories.

Towards a Grothendieck construction for Gray ω-functors

Let us end this introduction with a noteworthy by-product result that we obtain in this paper. As noted before, the Grothendieck construction of an ω-functor $F\colon I^{\mathrm{o}} \to \omega\text{-}\mathcal{C}at$ can be defined as (the dual of) a comma ω-category. In fact, this comma ω-category is (the dual of) a relative slice, meaning it is obtained by pulling back a slice ω-category as follows:

$$\begin{array}{ccc} (\int_I F)^{\mathrm{o}} & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathcal{C}at \\ \big\downarrow & \lrcorner & \big\downarrow \\ I^{\mathrm{o}} & \xrightarrow[F]{} & \omega\text{-}\mathcal{C}at \end{array} \quad .$$

The advantage of this description is that it can be adapted straightforwardly in the context of Gray ω-categories, up to some subtleties on dualities. If $\mathbb{C}$ is a (small) Gray ω-category, then its dual obtained by reversing the orientation of all the cells is not a Gray ω-category but what we call an *anti* Gray ω-category. By that, we mean a category enriched in ω-$\mathcal{C}at$, endowed with the oplax Gray tensor product, but with the composition morphisms of type

$$\underline{\mathsf{Hom}}(X,Y)\otimes\underline{\mathsf{Hom}}(Y,Z)\to\underline{\mathsf{Hom}}(X,Z)$$

(as opposed to $\underline{\mathsf{Hom}}(Y,Z)\otimes\underline{\mathsf{Hom}}(X,Y)\to\underline{\mathsf{Hom}}(X,Z)$, which is a different notion as the oplax Gray tensor product is not symmetrical and not even braided). As an example of an anti Gray ω-category, we have ω-$\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, whose 0-cells are strict ω-categories, 1-cells are strict ω-functors and higher cells are higher *lax* transformations. Now, given a (small) Gray ω-category $\mathbb{I}$ and $F\colon \mathbb{I}^{\mathrm{o}}\to\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ an anti Gray ω-functor, we define the (dual of the) Grothendieck construction $\int_{\mathbb{I}} F$ of F as the following pullback:

$$\begin{array}{ccc} (\int_{\mathbb{I}} F)^{\mathrm{o}} & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{I}^{\mathrm{o}} & \xrightarrow[F]{} & \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \end{array} \quad .$$

Note that $\int_{\mathbb{I}} F$ is indeed a Gray ω-category (and not an anti Gray ω-category). As already said, the dualities for strict and Gray ω-categories are subtle and we study them thoroughly in this paper.

We plan to work more extensively on this Grothendieck construction in the context of Gray ω-categories in future work.

Acknowledgments

The authors would like to thank Andrea Gagna, Tommy-Lee Klein, Félix Loubaton, Maxime Lucas and Georges Maltsiniotis for helpful conversations on the subject of this paper, as well as the referee for insightful suggestions that improved the paper.

1. Preliminaries on enriched categories

We begin with some preliminaries on categories enriched in a monoidal category. Our case of interest is the category of strict ω-categories endowed with the Gray tensor product, which will be introduced in the next section. This tensor product is *not* symmetric (and not even braided).

1.1. — Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category. Since we do not assume $\mathcal{V}$ to be symmetric, we need to distinguish between the notion of a $\mathcal{V}$-category (or a category enriched in $\mathcal{V}$) and that of an *anti* $\mathcal{V}$-category.

A *$\mathcal{V}$-category* $\mathcal{A}$ is given by

- a set of *objects* $\mathsf{Ob}(\mathcal{A})$,
- for all objects X and Y of $\mathcal{A}$, an *object of morphisms* $\underline{\mathsf{Hom}}_{\mathcal{A}}(X, Y)$ in $\mathcal{V}$,
- for all objects X, Y and Z of $\mathcal{A}$, a *composition morphism*
$$\circ\colon \underline{\mathsf{Hom}}_{\mathcal{A}}(Y, Z) \otimes \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Y) \to \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Z)$$
in $\mathcal{V}$,
- for every object X of $\mathcal{A}$, an *identity morphism*
$$1_X : I \to \underline{\mathsf{Hom}}_{\mathcal{A}}(X, X)$$
in $\mathcal{V}$,

satisfying well-known axioms.

The notion of an *anti* $\mathcal{V}$-category is obtained likewise but with composition morphisms of type
$$\circ\colon \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Y) \otimes \underline{\mathsf{Hom}}_{\mathcal{A}}(Y, Z) \to \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Z)\,.$$
More formally, an anti $\mathcal{V}$-category is a $\overline{\mathcal{V}}$-category, where $\overline{\mathcal{V}}$ denotes the monoidal category $(\mathcal{V}, \overline{\otimes}, I)$, the monoidal product $\overline{\otimes}$ being defined by $X \mathbin{\overline{\otimes}} Y = Y \otimes X$.

We denote by $\mathcal{V}\text{-}\mathcal{C}at$ the (large) 2-category whose objects are $\mathcal{V}$-categories, whose morphisms are $\mathcal{V}$-functors and whose 2-morphisms are $\mathcal{V}$-natural transformations. Consequently, the 2-category $\overline{\mathcal{V}}\text{-}\mathcal{C}at$ is the 2-category of anti $\mathcal{V}$-categories. Its morphisms are called *anti $\mathcal{V}$-functors* and its 2-morphisms *anti $\mathcal{V}$-natural transformations.*

1.2. — Let $\mathcal{V}$ be a monoidal category. Given a $\mathcal{V}$-category $\mathcal{A}$, we define its *transpose* to be the obvious *anti* $\mathcal{V}$-category $\mathcal{A}^{\mathrm{t}}$ which has the same set of objects as $\mathcal{A}$ and whose homs are defined by
$$\underline{\mathsf{Hom}}_{\mathcal{A}^{\mathrm{t}}}(X, Y) = \underline{\mathsf{Hom}}_{\mathcal{A}}(Y, X)\,.$$
In particular, applying this to $\overline{\mathcal{V}}$, we get that the transpose of an anti $\mathcal{V}$-category is a $\mathcal{V}$-category.

The correspondence that sends a $\mathcal{V}$-category to its transpose canonically extends to a 2-functor
$$\begin{aligned}(-)^{\mathrm{t}}\colon (\mathcal{V}\text{-}\mathcal{C}at)^{\mathrm{co}} &\to \overline{\mathcal{V}}\text{-}\mathcal{C}at\\ \mathcal{A} &\mapsto \mathcal{A}^{\mathrm{t}}\,,\end{aligned}$$
where the decoration "co" indicates that the orientation of the 2-cells of $\mathcal{V}\text{-}\mathcal{C}at$ is reversed.

1.3. — Let $\mathcal{V}$ be a monoidal category. Recall that if $\mathcal{V}$ admits limits indexed by a given category, then so does $\mathcal{V}\text{-}\mathcal{C}at$. In particular, let us describe the case of binary products explicitly. If $\mathcal{A}$ and $\mathcal{B}$ are two $\mathcal{V}$-categories, their product $\mathcal{V}$-category $\mathcal{A} \times \mathcal{B}$ can be described in the following way:

- $\mathsf{Ob}(\mathcal{A} \times \mathcal{B}) = \mathsf{Ob}(\mathcal{A}) \times \mathsf{Ob}(\mathcal{B})$,
- if X, X' are objects of $\mathcal{A}$ and Y, Y' are objects of $\mathcal{B}$, then
$$\underline{\mathsf{Hom}}_{\mathcal{A}\times\mathcal{B}}((X,Y),(X',Y')) = \underline{\mathsf{Hom}}_{\mathcal{A}}(X,X') \times \underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y')\,,$$
- if X, X', X'' are objects of $\mathcal{A}$ and Y, Y', Y'' are objects of $\mathcal{B}$, the composition of $\mathcal{A} \times \mathcal{B}$ is given by
$$\begin{array}{c}\big(\underline{\mathsf{Hom}}_{\mathcal{A}}(X',X'') \times \underline{\mathsf{Hom}}_{\mathcal{B}}(Y',Y'')\big) \otimes \big(\underline{\mathsf{Hom}}_{\mathcal{A}}(X,X') \times \underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y')\big) \\ \Big\downarrow{\scriptstyle (p_1\otimes p_1,\, p_2\otimes p_2)} \\ \big(\underline{\mathsf{Hom}}_{\mathcal{A}}(X',X'') \otimes \underline{\mathsf{Hom}}_{\mathcal{A}}(X,X')\big) \times \big(\underline{\mathsf{Hom}}_{\mathcal{B}}(Y',Y'') \otimes \underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y')\big) \\ \Big\downarrow{\scriptstyle \circ\times\circ} \\ \underline{\mathsf{Hom}}_{\mathcal{A}}(X,X'') \times \underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y'') \quad ,\end{array}$$
where p_1 and p_2 denote the first and second projections of the cartesian product,
- if X is an object of $\mathcal{A}$ and Y is an object of $\mathcal{B}$, the identity morphism of (X, Y) is
$$I \xrightarrow{(1_X,1_Y)} \underline{\mathsf{Hom}}_{\mathcal{A}}(X,X) \times \underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y)\,.$$

1.4. — Recall that if $F\colon \mathcal{V} \to \mathcal{V}'$ is a lax monoidal functor between monoidal categories, then F induces a 2-functor
$$F_*\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{V}'\text{-}\mathcal{C}at$$
sending a $\mathcal{V}$-category $\mathcal{A}$ to the obvious $\mathcal{V}'$-category $F_*(\mathcal{A})$ with the same set of objects as $\mathcal{A}$ and
$$\underline{\mathsf{Hom}}_{F_*(\mathcal{A})}(X,Y) = F(\underline{\mathsf{Hom}}_{\mathcal{A}}(X,Y))\,.$$

We call *monoidal functor* a lax monoidal functor whose structural natural transformations are isomorphisms. We say that a functor $F\colon \mathcal{V} \to \mathcal{V}'$ between two monoidal categories is *anti-monoidal* if it is monoidal considered as a functor $F\colon \overline{\mathcal{V}} \to \mathcal{V}'$, with the notation of 1.1.

1.5. — We say that a monoidal category $\mathcal{V}$ is *closed on the right* if, for every object Y of $\mathcal{V}$, the functor $-\otimes Y$ admits a right adjoint, then denoted by $\underline{\mathsf{Hom}}_r(Y,-)$. In this case, for X, Y and Z three objects of $\mathcal{V}$, we have a natural isomorphism
$$\mathsf{Hom}_{\mathcal{V}}(X \otimes Y, Z) \simeq \mathsf{Hom}_{\mathcal{V}}(X, \underline{\mathsf{Hom}}_r(Y,Z))\,.$$
If $\mathcal{V}$ is closed on the right, there is an obvious $\mathcal{V}$-category, denoted by $\mathcal{V}_r$, whose objects are the same as those of $\mathcal{V}$ and whose homs are given by the $\underline{\mathsf{Hom}}_r(X,Y)$.

Dually we say that $\mathcal{V}$ is *closed on the left* if, for every object X of $\mathcal{V}$, the functor $X \otimes -$ admits a right adjoint, denoted by $\underline{\mathsf{Hom}}_l(X, -)$. We then have a natural isomorphism
$$\mathsf{Hom}_{\mathcal{V}}(X \otimes Y, Z) \simeq \mathsf{Hom}_{\mathcal{V}}(Y, \underline{\mathsf{Hom}}_l(X, Z))\,.$$
If $\mathcal{V}$ is closed on the left, there is an obvious anti $\mathcal{V}$-category, denoted by $\mathcal{V}_l$, whose objects are the same as those of $\mathcal{V}$ and whose homs are given by the $\underline{\mathsf{Hom}}_l(X, Y)$.

We say that $\mathcal{V}$ is *biclosed* if it is closed both on the left and on the right. In this case, we have a canonical isomorphism
$$\underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}_r(Y, Z)) \simeq \underline{\mathsf{Hom}}_r(Y, \underline{\mathsf{Hom}}_l(X, Z))\,,$$
natural in X, Y and Z in $\mathcal{V}$.

1.6. — Let $\mathcal{V}$ be a monoidal category closed on the right. Suppose that $\mathcal{V}$ admits binary products. Then the product defines a $\mathcal{V}$-functor
$$\times\colon \mathcal{V}_r \times \mathcal{V}_r \to \mathcal{V}_r\,.$$
This $\mathcal{V}$-functor is given on objects by
$$(X, Y) \mapsto X \times Y\,,$$
and, if X, X', Y, Y' are four objects of $\mathcal{V}$, on homs by the morphism
$$\underline{\mathsf{Hom}}_r(X, X') \times \underline{\mathsf{Hom}}_r(Y, Y') \to \underline{\mathsf{Hom}}_r(X \times Y, X' \times Y')$$
obtained by adjunction from the composite
$$\begin{array}{l}\big(\underline{\mathsf{Hom}}_r(X, X') \times \underline{\mathsf{Hom}}_r(Y, Y')\big) \otimes \big(X \times Y\big) \\ \qquad\qquad {\scriptstyle (p_1 \otimes p_1,\, p_2 \otimes p_2)}\big\downarrow \\ \big(\underline{\mathsf{Hom}}_r(X, X') \otimes X\big) \times \big(\underline{\mathsf{Hom}}_r(Y, Y') \otimes Y\big) \xrightarrow{\mathrm{ev} \times \mathrm{ev}} X' \times Y'\,,\end{array}$$
where p_1 and p_2 denote the two projections of the product, and ev the evaluation morphism of the right internal hom.

Similarly, if $\mathcal{V}$ is closed on the left, we have a canonical anti $\mathcal{V}$-functor
$$\times\colon \mathcal{V}_l \times \mathcal{V}_l \to \mathcal{V}_l\,.$$

1.7. — Suppose now that $\mathcal{V}$ is a biclosed monoidal category and let $\mathcal{A}$ be a $\mathcal{V}$-category. A *copresheaf* over $\mathcal{A}$ is a $\mathcal{V}$-functor
$$F\colon \mathcal{A} \to \mathcal{V}_r\,.$$
We denote by $\underline{\mathsf{Hom}}(\mathcal{A}, \mathcal{V}_r)$ the category of copresheaves over $\mathcal{A}$ and $\mathcal{V}$-natural transformations between them.

A *presheaf* over $\mathcal{A}$ is an anti $\mathcal{V}$-functor
$$F\colon \mathcal{A}^{\mathrm{t}} \to \mathcal{V}_l\,.$$
We denote by $\underline{\mathsf{Hom}}(\mathcal{A}^{\mathrm{t}}, \mathcal{V}_l)$ the category of presheaves over $\mathcal{A}$ and anti $\mathcal{V}$-natural transformations between them.

Example 1.8. — Let a be an object of a $\mathcal{V}$-category $\mathcal{A}$. Then $\underline{\mathsf{Hom}}_{\mathcal{A}}(a,-)$ is a copresheaf over $\mathcal{A}$, and $\underline{\mathsf{Hom}}_{\mathcal{A}}(-,a)$ is a presheaf over $\mathcal{A}$.

As we shall now see, copresheaves and presheaves admit a useful description in terms of left and right modules.

1.9. — Let $\mathcal{V}$ be a biclosed monoidal category and let $\mathcal{A}$ be a $\mathcal{V}$-category. A *right $\mathcal{A}$-module* consists of

- a family $(F_a)_{a\in\mathsf{Ob}(\mathcal{A})}$ of objects of $\mathcal{V}$,
- for all objects a and a' of $\mathcal{A}$, a morphism of $\mathcal{V}$

$$\rho_{a,a'}\colon F_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')\to F_a\,,$$

such that

- for all objects a, a' and a'' of $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F_{a''}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a',a'')\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow{F_{a''}\otimes\circ} & F_{a''}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a'') \\ {\scriptstyle \rho_{a',a''}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')}\downarrow & & \downarrow{\scriptstyle \rho_{a,a''}} \\ F_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow[\rho_{a,a'}]{} & F_a \end{array}$$

  commutes,
- for every object a of $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F_a\simeq F_a\otimes I & \xrightarrow{F_a\otimes 1_a} & F_a\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a) \\ & \searrow^{=} & \downarrow{\scriptstyle \rho_{a,a}} \\ & & F_a \end{array}$$

  commutes.

If F and F' are two right $\mathcal{A}$-modules, a *morphism of right $\mathcal{A}$-modules* $u\colon F\to F'$ consists of a family $(u_a\colon F_a\to F'_a)_{a\in\mathsf{Ob}(\mathcal{A})}$ of morphisms of $\mathcal{V}$ such that for all objects a and a' of $\mathcal{A}$, the square

$$\begin{array}{ccc} F_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow{\rho_{a,a'}} & F_a \\ {\scriptstyle u_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')}\downarrow & & \downarrow{\scriptstyle u_a} \\ F'_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow[\rho'_{a,a'}]{} & F'_a \end{array}$$

commutes. Right $\mathcal{A}$-modules and their morphisms form a category that we will denote by $\mathrm{Mod}_{\mathcal{A}}$.

The notions of *left $\mathcal{A}$-module* and of *morphism of left $\mathcal{A}$-modules* are defined analogously but with an action morphism of type

$$\lambda_{a,a'}\colon F_a\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')\to F_{a'}\,.$$

We denote the category of left $\mathcal{A}$-modules by ${}_{\mathcal{A}}\mathrm{Mod}$.

Proposition 1.10. — *Let $\mathcal{V}$ be a biclosed monoidal category and let $\mathcal{A}$ be a $\mathcal{V}$-category. We have isomorphisms of categories*

$$ {}_{\mathcal{A}}\mathrm{Mod} \xrightarrow{\sim} \underline{\mathsf{Hom}}(\mathcal{A}, \mathcal{V}_r)\,, $$

and

$$ \mathrm{Mod}_{\mathcal{A}} \xrightarrow{\sim} \underline{\mathsf{Hom}}(\mathcal{A}^{\mathrm{t}}, \mathcal{V}_l)\,. $$

Proof. — This follows from the adjunctions between the tensor product and $\underline{\mathsf{Hom}}_r$ and $\underline{\mathsf{Hom}}_l$. We leave the details to the reader. □

Remark 1.11. — In practice, in this paper, we will produce presheaves and then use the previous proposition to obtain right modules and do computations using the laws of right modules.

As we shall see in the next section, one specific property of the Gray tensor product is that its monoidal unit is the terminal object. We now develop some enriched category theory with this additional hypothesis.

1.12. — A monoidal category $(\mathcal{V}, \otimes, I)$ is said to *have projections* if $\mathcal{V}$ admits finite products and the tensor unit of $\mathcal{V}$ is a terminal object. In this case, if X and Y are two objects of $\mathcal{V}$, we get "projections"

$$ X \xleftarrow{\sim} X \otimes I \xleftarrow{X \otimes p_Y} X \otimes Y \xrightarrow{p_X \otimes Y} I \otimes Y \xrightarrow{\sim} Y \quad , $$

where $p_Z\colon Z \to I$ denotes the unique morphism to the terminal object. In particular, we get a morphism

$$ \pi = (\pi_1, \pi_2)\colon X \otimes Y \to X \times Y\,, $$

natural in X and Y in $\mathcal{V}$.

The cartesian product defines a monoidal structure on $\mathcal{V}$ and we will denote by $\mathcal{V}^{\times}$ the resulting monoidal category. By default, $\mathcal{V}$ will be endowed with the monoidal product $\otimes$ but sometimes, to emphasize this, we will denote this monoidal category by $\mathcal{V}^{\otimes}$.

With this notation, the morphism π shows that the identity functor of $\mathcal{V}$ is a lax monoidal functor from $\mathcal{V}^{\times}$ to $\mathcal{V}^{\otimes}$. It is also lax monoidal considered with values in $\overline{\mathcal{V}^{\otimes}}$. In particular, we get 2-functors

$$ \mathcal{V}^{\times}\text{-}\mathcal{C}at \to \mathcal{V}^{\otimes}\text{-}\mathcal{C}at \quad \text{and} \quad \mathcal{V}^{\times}\text{-}\mathcal{C}at \to \overline{\mathcal{V}^{\otimes}}\text{-}\mathcal{C}at\,. $$

If the morphism $\pi\colon X \otimes Y \to X \times Y$ is an epimorphism for all objects X and Y of $\mathcal{V}$, we will say that $\mathcal{V}$ has *jointly surjective projections*. In this case, the two 2-functors above are injective on objects and fully faithful.

Suppose moreover that $\mathcal{V}$ is cartesian closed and that $\mathcal{V}^{\otimes}$ is monoidal biclosed. We will denote by $\mathcal{V}_{\mathrm{cart}}$ the $\mathcal{V}^{\times}$-category obtained by enriching $\mathcal{V}$ over itself using the cartesian internal hom. In other words, if X and Y are two objects of $\mathcal{V}$, then $\underline{\mathsf{Hom}}_{\mathcal{V}_{\mathrm{cart}}}(X, Y) = \underline{\mathsf{Hom}}(X, Y)$, where $\underline{\mathsf{Hom}}$ denotes the cartesian internal hom of $\mathcal{V}$. Based on the above, $\mathcal{V}_{\mathrm{cart}}$ can also be considered as either a $\mathcal{V}^{\otimes}$-category or a $\overline{\mathcal{V}^{\otimes}}$-category. By the Yoneda lemma, using the morphism π, we get canonical morphisms

$$\underline{\mathsf{Hom}}_{\mathcal{V}_{\mathrm{cart}}}(X,Y) \to \underline{\mathsf{Hom}}_r(X,Y) \quad \text{and} \quad \underline{\mathsf{Hom}}_{\mathcal{V}_{\mathrm{cart}}}(X,Y) \to \underline{\mathsf{Hom}}_l(X,Y)\,.$$

These morphisms are monomorphisms if $\mathcal{V}$ has jointly surjective projections. In any case, they induce a $\mathcal{V}^{\otimes}$-functor and a $\overline{\mathcal{V}^{\otimes}}$-functor

$$\mathcal{V}_{\mathrm{cart}} \to \mathcal{V}_r \quad \text{and} \quad \mathcal{V}_{\mathrm{cart}} \to \mathcal{V}_l\,.$$

1.13. — Let $\mathcal{V}$ be a monoidal category with projections. If X, Y and Z are three objects of $\mathcal{V}$, we have a canonical natural morphism

$$\varphi\colon X \otimes (Y \times Z) \to (X \otimes Y) \times Z\,,$$

given on components by

$$X \otimes (Y \times Z) \xrightarrow{X \otimes p_1} X \otimes Y \quad \text{and} \quad X \otimes (Y \times Z) \xrightarrow{\pi_2} Y \times Z \xrightarrow{p_2} Z\,,$$

where p_1 and p_2 denote the projections of the cartesian product, and π_1 and π_2 the "projections" of the tensor product. We will not need this but one can actually show that φ is a tensorial strength on the functor $- \times Z$.

Suppose moreover that $\mathcal{V}$ is cartesian closed and monoidal biclosed. Then the morphism φ induces a natural morphism

$$\lambda\colon\ \underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y,Z)) \to \underline{\mathsf{Hom}}(Y, \underline{\mathsf{Hom}}_l(X,Z))$$

that makes the following square commutative

$$\begin{array}{ccc}
\underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y,Z)) & \xrightarrow{\lambda} & \underline{\mathsf{Hom}}(Y, \underline{\mathsf{Hom}}_l(X,Z)) \\
\downarrow & & \downarrow \\
\underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}_r(Y,Z)) & \xrightarrow{\sim} & \underline{\mathsf{Hom}}_r(Y, \underline{\mathsf{Hom}}_l(X,Z))
\end{array}\quad,$$

where the vertical arrows are induced by the canonical morphism from $\underline{\mathsf{Hom}}$ to $\underline{\mathsf{Hom}}_r$. Explicitly, the morphism λ is obtained by adjunction from

$$\begin{array}{l}
X \otimes \big(\underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y,Z)) \times Y\big) \\
\qquad\qquad {\scriptstyle\varphi}\downarrow \\
\big(X \otimes \underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y,Z))\big) \times Y \xrightarrow{\mathrm{ev} \times Y} \underline{\mathsf{Hom}}(Y,Z) \times Y \xrightarrow{\mathrm{ev}} Z
\end{array}\quad,$$

where ev denotes the evaluation morphism. It follows from the commutative square above that if $\mathcal{V}$ has jointly surjective projections, then λ is a monomorphism.

Remark 1.14. — The previous paragraph can be dualized to $\underline{\mathsf{Hom}}_r$. In particular, there is a canonical natural morphism

$$\underline{\mathsf{Hom}}_r(X, \underline{\mathsf{Hom}}(Y,Z)) \to \underline{\mathsf{Hom}}(Y, \underline{\mathsf{Hom}}_r(X,Z))\,.$$

2. Preliminaries on strict ω-categories and Gray ω-categories

2.1. — For any $n \geqslant 1$, we denote by $n\text{-}\mathcal{C}at$ the category of (small) strict n-categories and strict n-functors, that is, the category of categories enriched in $(n-1)\text{-}\mathcal{C}at$ (with the cartesian monoidal structure), the category $0\text{-}\mathcal{C}at$ being the category $\mathcal{S}et$ of sets.

We have a canonical inclusion

$$(n-1)\text{-}\mathcal{C}at \hookrightarrow n\text{-}\mathcal{C}at$$

sending a strict $(n-1)$-category to the strict n-category obtained by adding only trivial n-cells (that is, identities). This inclusion admits a right adjoint

$$\tau_{n-1} \colon n\text{-}\mathcal{C}at \to (n-1)\text{-}\mathcal{C}at\,,$$

and the category $\omega\text{-}\mathcal{C}at$ of strict ω-categories and strict ω-functors is obtained as the limit of the diagram

$$\cdots \longrightarrow 2\text{-}\mathcal{C}at \xrightarrow{\tau_1} 1\text{-}\mathcal{C}at \xrightarrow{\tau_0} 0\text{-}\mathcal{C}at \quad .$$

For any $n \geqslant 0$, we have a canonical fully faithful functor

$$n\text{-}\mathcal{C}at \hookrightarrow \omega\text{-}\mathcal{C}at\,,$$

admitting both a left and a right adjoint, and whose image consists exactly of those strict ω-categories with only trivial k-cells for $k > n$. We shall always consider the previous fully faithful functor as an inclusion.

From now on, we will drop the adjective "strict" and simply refer to "strict ω-categories" and "strict ω-functors" as "ω-categories" and "ω-functors" (and similarly for "n-categories" and "n-functors").

Let us introduce some notation.

2.2. — Let C be an ω-category. If x is a k-cell of C, for $0 \leqslant i \leqslant k$, we denote by

$$s_i(x) \quad \text{and} \quad t_i(x)$$

the i-dimensional source and target of x, also called i-source and i-target of x. In the case where $i = k-1$, we also write $s(x)$ and $t(x)$. For $i \geqslant k$, we denote by

$$1^i_x$$

the i-dimensional unit of x. When $i = k+1$, we also write 1_x.

Given two k-cells x and y such that $s_i(x) = t_i(y)$, with $i < k$, we denote by

$$x *_i y$$

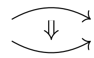
their i-composition. More generally, for $k > i$ and $l > i$, x a k-cell and y an l-cell such that $s_i(x) = t_i(y)$, then if $k \leqslant l$, we set

$$x *_i y = 1^l_x *_i y\,,$$

and if $k \geqslant l$, we set

$$x *_i y = x *_i 1^l_y\,.$$

2.3. — For any $n \geqslant 0$, we define D_n to be the n-category freely generated by a generic n-cell. Explicitly, D_n has exactly one non-trivial n-cell e_n, and exactly two non-trivial k-cells for $0 \leqslant k < n$ given by $s_k(e_n)$ and $t_k(e_n)$.

$$\mathrm{D}_0 = \bullet\ , \quad \mathrm{D}_1 = \bullet \longrightarrow \bullet\ , \quad \mathrm{D}_2 = \bullet$$  $$\bullet\ , \quad \cdots$$

In particular, D_n represents the functor $\omega\text{-}\mathcal{C}at \to \mathcal{S}et$ that sends an ω-category to its set of n-cells. If x is an n-cell of an ω-category C, we will denote by $\tilde{x} \colon \mathrm{D}_n \to C$ the corresponding ω-functor.

2.4. — The category $\omega\text{-}\mathcal{C}at$ has several interesting monoidal structures. First, it is cartesian closed and the associated internal hom will be denoted by

$$\underline{\mathsf{Hom}}(A, B)\,,$$

for A and B two ω-categories. The n-cells of this ω-category are in bijection with the ω-functors

$$\mathrm{D}_n \times A \to B\,.$$

In particular, the 0-cells are simply the ω-functors $A \to B$. For $n \geqslant 1$, the n-cells are referred to as *strict n-transformations*. Explicitly, given two ω-functors $u, v \colon A \to B$, a strict n-transformation α, with 0-source u and 0-target v, is a family (α_x) of n-cells of B, indexed by the 0-cells x of A, such that

– for every 0-cell x of A, we have

$$s_0(\alpha_x) = u(x) \quad \text{and} \quad t_0(\alpha_x) = v(x)\,,$$

– for every k-cell x of A, with $k \geqslant 1$, we have

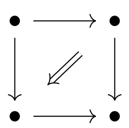
$$\alpha_{t_0(x)} *_0 u(x) = v(x) *_0 \alpha_{s_0(x)}\,.$$

The category $\omega\text{-}\mathcal{C}at$ is enriched over itself via the cartesian product, and we have a "fixed point" property: the category of categories enriched in $(\omega\text{-}\mathcal{C}at, \times, \mathrm{D}_0)$ is canonically isomorphic to $\omega\text{-}\mathcal{C}at$ itself. In particular, we have a (large) ω-category

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

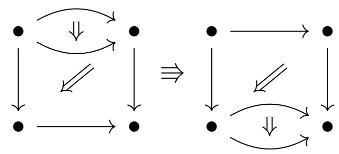
whose 0-cells are (small) ω-categories, whose 1-cells are ω-functors and whose n-cells, with $n > 1$, are strict $(n-1)$-transformations.

2.5. — Another fundamental monoidal structure on $\omega\text{-}\mathcal{C}at$ comes from the so-called *(oplax) Gray tensor product* (see for example [**4**, Appendix A]), denoted by $\otimes$. To give an intuition, the tensor product $\mathrm{D}_1 \otimes \mathrm{D}_1$ is a square with a non-trivial 2-cell

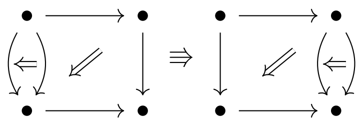


,

whereas the cartesian product $\mathrm{D}_1 \times \mathrm{D}_1$ is a commutative square. The monoidal unit is the terminal ω-category D_0 (as for the cartesian structure) and the Gray tensor product thus defines a monoidal structure with projections in the sense of 1.12. This monoidal structure is *not* symmetrical (and not even braided). For example, we have

$\mathrm{D}_1 \otimes \mathrm{D}_2 =$ 

and

$\mathrm{D}_2 \otimes \mathrm{D}_1 =$  .

2.6. — Let A and B be two ω-categories. If x is a k-cell of A and y an l-cell of B, then there is an associated $(k+l)$-cell $x \otimes y$ in $A \otimes B$. Explicitly, with the notation of 2.3, this cell corresponds to the ω-functor

$$\mathrm{D}_{k+l} \xrightarrow{\tilde{c}} \mathrm{D}_k \otimes \mathrm{D}_l \xrightarrow{\tilde{x} \otimes \tilde{y}} A \otimes B\,,$$

where c denotes the *principal cell* of $\mathrm{D}_k \otimes \mathrm{D}_l$, that is, its unique non-trivial $(k+l)$-cell (see for instance [**3**, paragraph B.1.5]).

We will not need this but one can show that cells of the form $x \otimes y$ generate $A \otimes B$ under composition.

2.7. — The Gray tensor product is biclosed, with right and left internal homs denoted by $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}$, respectively, so that we have

$$\begin{aligned} \mathsf{Hom}(A \otimes B, C) &\simeq \mathsf{Hom}(A, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B, C)) \\ &\simeq \mathsf{Hom}(B, \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, C))\,, \end{aligned}$$

for A, B and C three ω-categories. Moreover, by 1.5, this last bijection can be promoted to a natural isomorphism

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B, C)) \simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B, \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, C))\,.$$

The n-cells of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ (resp. of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$) are in bijection with the ω-functors

$$\mathrm{D}_n \otimes A \to B \quad (\text{resp. } A \otimes \mathrm{D}_n \to B)\,.$$

In particular, the 0-cells of both $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$ are simply the ω-functors $A \to B$. For $n \geqslant 1$, an n-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ (resp. of $\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$) is called an *oplax n-transformation* (resp. a *lax n-transformation*). For $n = 1$, we will simply say *oplax transformations* (resp. *lax transformations*). For an explicit description of oplax transformations, see for example [**4**, paragraph 1.9].

2.8. — By definition, if $u, v\colon A \to B$ are two ω-functors, an oplax transformation α from u to v corresponds to an ω-functor $h\colon \mathrm{D}_1 \otimes A \to B$ making the following diagram commutative

$$\begin{array}{ccc}
\mathrm{D}_0 \otimes A \simeq A & \overset{u}{\searrow} & \\
\Big\downarrow{\scriptstyle \sigma \otimes A} & & \\
\mathrm{D}_1 \otimes A & \xrightarrow{\ h\ } & A \\
\Big\uparrow{\scriptstyle \tau \otimes A} & & \\
\mathrm{D}_0 \otimes A \simeq A & \underset{v}{\nearrow} &
\end{array},$$

where $\sigma, \tau\colon \mathrm{D}_0 \to \mathrm{D}_1$ correspond to the source and the target of the non-trivial 1-cell of D_1. By adjunction, it also corresponds to an ω-functor $k\colon A \to \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B)$ making the diagram

$$\begin{array}{ccc}
 & \overset{u}{\nearrow} & B \simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, B) \\
 & & \Big\uparrow{\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{lax}}(\sigma, B)} \\
A & \xrightarrow{\ k\ } & \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B) \\
 & & \Big\downarrow{\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{lax}}(\tau, B)} \\
 & \underset{v}{\searrow} & B \simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, B)
\end{array}$$

commute. This leads us to set

$$\Gamma B = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B)\,.$$

This ω-category is the *ω-category of cylinders in B*. A k-cell in this ω-category is called a *k-cylinder in B*. In other words, a k-cylinder in B is an ω-functor $\mathrm{D}_1 \otimes \mathrm{D}_k \to B$. If $\beta\colon \mathrm{D}_1 \otimes \mathrm{D}_k \to B$ is such a k-cylinder, the image of the principal cell of $\mathrm{D}_1 \otimes \mathrm{D}_k$ (see 2.6) in B will be called the *principal cell* of β. We will denote it by β_k.

As a particular case of the compatibilities between $\underline{\mathsf{Hom}}_{\mathrm{lax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ (see 2.7), we get that if A and B are two ω-categories, then we have a natural isomorphism

$$\Gamma\,\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B) \simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, \Gamma B)\,.$$

This isomorphism will play an important role in this paper.

In [**6**, Section 4] (see also [**7**, Appendix A]), the authors describe the ω-category ΓC inductively. The two following paragraphs provide preliminaries to phrase their description.

2.9. — If C is an ω-category, then the ω-category ΓC is naturally the object of morphisms of a category internal to ω-$\mathcal{C}at$. Indeed, the functors

$$\sigma, \tau\colon \mathrm{D}_0 \to \mathrm{D}_1, \quad \kappa\colon \mathrm{D}_1 \to \mathrm{D}_0 \quad \text{and} \quad \nabla\colon \mathrm{D}_1 \to \mathrm{D}_1 \amalg_{\mathrm{D}_0} \mathrm{D}_1\,,$$

corresponding respectively to the source and the target of the unique non-trivial 1-cell of D_1, the unit of the unique object of D_0 and the total composition of $\mathrm{D}_1 \amalg_{\mathrm{D}_0} \mathrm{D}_1$, define a cocategory internal to categories, and hence internal to ω-categories. By applying the functor $\underline{\mathsf{Hom}}_{\mathrm{lax}}(-, C)$ which sends colimits to limits, we get ω-functors

$$\mathbb{s}, \mathbb{t} \colon \Gamma C \to C, \quad \mathbb{k} \colon C \to \Gamma C \quad \text{and} \quad *_c \colon \Gamma C \times_C \Gamma C \to \Gamma C$$

defining a structure of category internal to ω-categories. If x is a cell of C, we will denote $\mathbb{k}(x)$ by $\mathbb{1}_x$.

2.10. — Let C be an ω-category and let c and d be two objects of C. For every 1-cell $u \colon c' \to c$, we have an ω-functor

$$\Gamma \underline{\mathsf{Hom}}_C(u, d) \colon \Gamma \underline{\mathsf{Hom}}_C(c, d) \to \Gamma \underline{\mathsf{Hom}}_C(c', d)\,.$$

If α is a cell in $\Gamma \underline{\mathsf{Hom}}_C(c, d)$, we will denote its image under this ω-functor by $\alpha *_r u$.

More generally, as the ω-functor $\Gamma = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, -)$ commutes with limits, the composition ω-functor

$$\underline{\mathsf{Hom}}_C(c, d) \times \underline{\mathsf{Hom}}_C(c', c) \to \underline{\mathsf{Hom}}_C(c', d)$$

induces an ω-functor

$$\Gamma \underline{\mathsf{Hom}}_C(c, d) \times \Gamma \underline{\mathsf{Hom}}_C(c', c) \to \Gamma \underline{\mathsf{Hom}}_C(c', d)\,,$$

and if α is a k-cell of $\Gamma \underline{\mathsf{Hom}}_C(c, d)$ and u a k-cell of $\underline{\mathsf{Hom}}_C(c', c)$, we define $\alpha *_r u$ to be the image of the pair $(\alpha, \mathbb{1}_u)$ by this ω-functor.

Similarly, if $v \colon d \to d'$ is a 1-cell of C, we have an ω-functor

$$\Gamma \underline{\mathsf{Hom}}_C(c, v) \colon \Gamma \underline{\mathsf{Hom}}_C(c, d) \to \Gamma \underline{\mathsf{Hom}}_C(c, d')\,,$$

and if α is a cell in $\Gamma \underline{\mathsf{Hom}}_C(c, d)$, its image will be denoted by $v *_l \alpha$. More generally, if v is a k-cell of $\underline{\mathsf{Hom}}_C(d, d')$ and α is a k-cell of $\Gamma \underline{\mathsf{Hom}}_C(c, d)$, we define $v *_l \alpha$ to be the image of the pair $(\mathbb{1}_v, \alpha)$ by the ω-functor

$$\Gamma \underline{\mathsf{Hom}}_C(d, d') \times \Gamma \underline{\mathsf{Hom}}_C(c, d) \to \Gamma \underline{\mathsf{Hom}}_C(c, d')\,.$$

Remark 2.11. — We will come back to these operations in terms of modules in 4.2 and Remark 4.7.

2.12. — Let C be an ω-category. By [**6**, Section 4], the ω-category ΓC can be described (up to isomorphism) as a category enriched in ω-categories in the following way:

- The objects of ΓC are the 1-cells of C.
- If $f \colon c \to d$ and $f' \colon c' \to d'$ are two objects of ΓC, we have

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\Gamma C}(f, f')\\ &\qquad = \underline{\mathsf{Hom}}_C(c, c') \times_{\underline{\mathsf{Hom}}_C(c,d')} \Gamma\underline{\mathsf{Hom}}_C(c, d') \times_{\underline{\mathsf{Hom}}_C(c,d')} \underline{\mathsf{Hom}}_C(d, d')\,,\end{aligned}$$

where this iterated fiber product denotes the limit of the diagram

$$\begin{array}{ccccccc}
\underline{\mathsf{Hom}}_C(c,c') & & & \Gamma\underline{\mathsf{Hom}}_C(c,d') & & & \underline{\mathsf{Hom}}_C(d,d') \\
& \searrow^{f'*_0-} & \swarrow_{\mathbb{s}} & & \searrow^{\mathbb{t}} & \swarrow_{-*_0 f} & \\
& \underline{\mathsf{Hom}}_C(c,d') & & & & \underline{\mathsf{Hom}}_C(c,d') &
\end{array} .$$

Concretely, a k-cell in this hom is a triple (u, α, v) in

$$\underline{\mathsf{Hom}}_C(c,c')_k \times \Gamma\underline{\mathsf{Hom}}_C(c,d')_k \times \underline{\mathsf{Hom}}_C(d,d')_k$$

such that

$$\mathbb{s}(\alpha) = f' *_0 u \quad \text{and} \quad \mathbb{t}(\alpha) = v *_0 f\,. \tag{$*$}$$

This formula is an ω-categorification of the formula for a 1-cylinder, i.e., a 2-square:

$$\begin{array}{ccc}
c & \xrightarrow{u} & c' \\
{\scriptstyle f}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle f'} \\
d & \xrightarrow[v]{} & d'
\end{array} .$$

In other words, a $(k+1)$-cylinder in C is given by its 0-source $f\colon c \to d$ and its 0-target $f'\colon c' \to d'$, a $(k+1)$-cell u in C of 0-source c and 0-target c', a $(k+1)$-cell v in C of 0-source d and 0-target d', and a k-cylinder α in $\underline{\mathsf{Hom}}_C(c,d')$ satisfying the relations $(*)$.

- If f, f', f'' are three objects of ΓC, the composition ω-functor

$$\underline{\mathsf{Hom}}_{\Gamma C}(f',f'') \times \underline{\mathsf{Hom}}_{\Gamma C}(f,f') \to \underline{\mathsf{Hom}}_{\Gamma C}(f,f'')$$

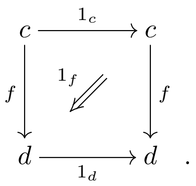
is given by

$$((u',\alpha',v'),(u,\alpha,v)) \mapsto (u' *_0 u, (v' *_l \alpha) *_c (\alpha' *_r u), v' *_0 v)\,.$$

This formula is an ω-categorification of the formula for composing the diagram

$$\begin{array}{ccccc}
c & \xrightarrow{u} & c' & \xrightarrow{u'} & c'' \\
{\scriptstyle f}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle f'} & \overset{\alpha'}{\swarrow} & \downarrow{\scriptstyle f''} \\
d & \xrightarrow[v]{} & d' & \xrightarrow[v']{} & d''
\end{array} .$$

– If $f\colon c \to d$ is an object of ΓC, its unit is the 1-cylinder $(1_c, \mathbb{1}_f, 1_d)$, corresponding to the commutative square



We now come to one of the central notions of this paper, the notion of a Gray ω-category, introduced in [**4**, Appendix C].

2.13. — A *Gray ω-category* is a $\mathcal{V}$-category for $\mathcal{V} = (\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$ and an *anti Gray ω-category* an anti $\mathcal{V}$-category, again for $\mathcal{V} = (\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$.

If $\mathbb{C}$ is a Gray ω-category, its objects will also be called 0-*cells*. If c and d are two 0-cells of $\mathbb{C}$, the k-cells of the ω-category $\underline{\mathsf{Hom}}_{\mathbb{C}}(c, d)$ will be called $(k+1)$-*cells* of $\mathbb{C}$. By definition, if x is such a cell, the 0-*source* $s_0(x)$ of x is c and its 0-*target* $t_0(x)$ is d. The composition $x *_i y$ of two k-cells of $\underline{\mathsf{Hom}}_{\mathbb{C}}(c, d)$ will be denoted by $x *_{i+1} y$ if the cells x and y are considered as $(k+1)$-cells of $\mathbb{C}$.

If a, b and c are three 0-cells of $\mathbb{C}$, the composition ω-functor will be denoted by

$$*_0\colon\ \underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(a, c)\,.$$

If x is a k-cell of 0-source b and 0-target c and y is an l-cell of 0-source a and 0-target b, we will denote by $x *_0 y$ the cell obtained by applying the composition ω-functor to the $(k+l-2)$-cell $x \otimes y$ of $\underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b)$. This cell is a $(k+l-2)$-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}}(a, c)$. In other words, $x *_0 y$ is a $(k+l-1)$-cell of $\mathbb{C}$. Its 0-source is a and its 0-target c. Note that the composition ω-functor is uniquely determined by all the $x *_0 y$.

In particular, the composition $*_0$ of a k-cell and a 1-cell of $\mathbb{C}$ is a k-cell. The composition $x *_0 y$ of two 2-cells of $\mathbb{C}$ is a 3-cell. In general, the exchange rule in $\mathbb{C}$ for two such 2-cells x and y does not hold on the nose but up to the non-invertible 3-cell $x *_0 y$:

$$(x *_0 t(y)) *_1 (s(x) *_0 y) \overset{x *_0 y}{\Longrightarrow} (t(x) *_0 y) *_1 (\beta *_0 s(y))$$

(see [**3**, Proposition B.1.14]). We refer the reader to [**3**, Section B.1] for more details on the concrete description of the structure of a Gray ω-category.

Similar definitions and notation apply to anti Gray ω-categories.

2.14. — Enriched functors between Gray ω-categories will be called *Gray ω-functors*. Similarly, enriched functors between anti Gray ω-categories will be called *anti Gray ω-functors*. A Gray or anti Gray ω-functor is uniquely determined by its action on cells. More precisely, morphisms between Gray or anti Gray ω-categories can be described as functions on k-cells for every $k \geqslant 0$ that are compatible with sources, targets, compositions and units.

Enriched natural transformations between Gray or anti Gray ω-functors will be called *strict transformations*. Explicitly, if $F, G\colon \mathbb{C} \to \mathbb{C}'$ are two Gray ω-functors (or two anti Gray ω-functors), a strict transformation $\alpha\colon F \Rightarrow G$ consists of the data of a 1-cell

$$\alpha_c\colon F(c) \to G(c)$$

for every 0-cell c of $\mathbb{C}$, such that, for every k-cell x of $\mathbb{C}$, with $k \geqslant 1$, we have

$$\alpha_{t_0(x)} *_0 F(x) = G(x) *_0 \alpha_{s_0(x)}\,.$$

2.15. — We will denote by

$$\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \quad (\text{resp. } \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})$$

the Gray ω-category (resp. the anti Gray ω-category) whose objects are ω-categories and whose homs are

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B) \quad (\text{resp. } \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B))\,,$$

for A and B are two ω-categories. By definition, the 0-cells of $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ (resp. of $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$) are ω-categories, its 1-cells are ω-functors, its 2-cells are oplax (resp. lax) transformations and, for $n > 2$, its n-cells are oplax (resp. lax) $(n-1)$-transformations.

2.16. — Since the monoidal unit of the Gray tensor product is the terminal ω-category, by 1.12, for all ω-categories A and B, we have a canonical ω-functor

$$\pi = (\pi_1, \pi_2)\colon A \otimes B \to A \times B\,.$$

Proposition 2.17. — *If A and B are two ω-categories, the ω-functor*

$$\pi\colon A \otimes B \to A \times B$$

is an epimorphism.

Proof. — The ω-category $A \times B$ is generated under composition by cells of the form $(x, 1^k_b)$, where x is a k-cell of A, with $k \geqslant 0$, and b a 0-cell of B, and of the form $(1^l_a, y)$, where a is a 0-cell of A and y an l-cell of B, with $l \geqslant 0$. It thus suffices to show that these cells are in the image of the ω-functor π. But if b is a 0-cell of B, considering the commutative diagram

$$\begin{array}{ccc} A \otimes \mathrm{D}_0 & \xrightarrow{A \otimes b} & A \otimes B \\ {\wr}\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ A \times \mathrm{D}_0 & \xrightarrow[A \times b]{} & A \times B \end{array}\quad,$$

we get that for every k-cell x of A, the cell $(x, 1_b^k)$ is in the image of $\pi\colon A \otimes B \to A \times B$. A similar argument shows that cells of the form $(1_a^l, y)$ are in the image of π, thereby proving the result. ☐

Remark 2.18. — We will not need this but one can actually prove that the ω-functor π of the proposition is surjective on cells. More precisely, if x is a k-cell of A and y is an l-cell of B, then we have $\pi(x \otimes y) = (1_x^{k+l}, 1_y^{k+l})$.

2.19. — In the language of 1.12, the previous proposition states that the monoidal category $(\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$ has jointly surjective projections. We can thus apply the considerations of 1.12 and 1.13. Let us start by 1.12.

We get that the category of ω-categories embeds both in the category of Gray ω-categories and in the category of anti Gray ω-categories. Moreover, we have canonical monomorphisms

$$\underline{\mathsf{Hom}}(A, B) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B) \quad \text{and} \quad \underline{\mathsf{Hom}}(A, B) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)\,,$$

which we will treat as inclusions.

We thus have canonical Gray and anti Gray ω-functors

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \quad \text{and} \quad \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,,$$

which are the identity on objects and faithful.

2.20. — Let us now apply 1.13 to $(\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$. If D, A, B are three ω-categories, we have a canonical monomorphism

$$\lambda\colon \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}(A, B)) \hookrightarrow \underline{\mathsf{Hom}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B))$$

that makes the square

$$\begin{array}{ccc} \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}(A, B)) & \xhookrightarrow{\lambda} & \underline{\mathsf{Hom}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B)) \\ \big\downarrow & & \big\downarrow \\ \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)) & \xrightarrow{\sim} & \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B)) \end{array}$$

commute. We will treat λ as an inclusion. We thus get a factorization

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}(A, B)) \hookrightarrow \underline{\mathsf{Hom}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B)) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B))$$

of the canonical inclusion. This applies in particular to the case where $D = \mathrm{D}_1$ in which we get inclusions

$$\Gamma\, \underline{\mathsf{Hom}}(A, B) \hookrightarrow \underline{\mathsf{Hom}}(A, \Gamma B) \hookrightarrow \Gamma\, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)\,.$$

Remark 2.21. — By adjunction, the k-cells of

$$\Gamma\, \underline{\mathsf{Hom}}(A, B), \quad \underline{\mathsf{Hom}}(A, \Gamma B) \quad \text{and} \quad \Gamma\, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$$

correspond to ω-functors from

$$(\mathrm{D}_1 \otimes \mathrm{D}_k) \times A, \quad \mathrm{D}_1 \otimes (\mathrm{D}_k \times A) \quad \text{and} \quad \mathrm{D}_1 \otimes \mathrm{D}_k \otimes A\,,$$

respectively, to B. In particular, for $A = \mathrm{D}_1$, the 1-cells correspond to cubes in B of shapes

$$(\mathrm{D}_1 \otimes \mathrm{D}_1) \times \mathrm{D}_1, \quad \mathrm{D}_1 \otimes (\mathrm{D}_1 \times \mathrm{D}_1) \quad \text{and} \quad \mathrm{D}_1 \otimes \mathrm{D}_1 \otimes \mathrm{D}_1\,,$$

respectively. This means that the 1-cells of $\Gamma\,\underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1, B)$ are fully lax cubes in B, those of $\underline{\mathsf{Hom}}(\mathrm{D}_1, \Gamma B)$ are commutative cubes with only four lax faces (the two commutative faces being opposite to each other) and those of $\Gamma\,\underline{\mathsf{Hom}}(\mathrm{D}_1, B)$ are commutative cubes with only two (opposite) lax faces.

We end the section by some considerations on the dualities of $\omega\text{-}\mathcal{C}at$.

2.22. — If $S \subset \mathbb{N}^*$ is a subset of the set of positive integers, then we will denote by

$$D_S\colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$$

the ω-functor sending an ω-category C to the ω-category obtained from C by reversing the orientation of all the cells whose dimension belongs to S. It is immediate that D_S is an involutive endofunctor of $\omega\text{-}\mathcal{C}at$. Actually, up to isomorphism, all the autoequivalences of $\omega\text{-}\mathcal{C}at$ are of the form D_S. We will sometimes refer to these autoequivalences as *dualities*.

Several special cases play an important role in the theory of $\omega\text{-}\mathcal{C}at$:

- If $S = \mathbb{N}^*$, then $D_{\mathbb{N}^*}$ is denoted by D_{o} and is called the *total dual*. We simply write C^{o} for the total dual of an ω-category C.
- If $S = 2\mathbb{N}+1$ is the set of odd integers, then $D_{2\mathbb{N}+1}$ is denoted by D_{op} and is called the *odd dual*. We simply write C^{op} for the odd dual of an ω-category C.
- If $S = 2\mathbb{N}^*$ is the set of positive even integers, then $D_{2\mathbb{N}^*}$ is denoted by D_{co} and is called the *even dual*. We simply write C^{co} for the odd dual of an ω-category C.
- If $S = \{1\}$, then $D_{\{1\}}$ is denoted by D_{t} and is called the *transpose*. We simply write C^{t} for the transpose of C. This coincides with the transpose of C in the sense of 1.2 when C is considered as a category enriched over $\omega\text{-}\mathcal{C}at$ endowed with the cartesian product.

By composing all these special dualities, we get a group of eight dualities. In particular, if C is an ω-category, we get seven other ω-categories

$$C^{\mathrm{o}}, \quad C^{\mathrm{op}}, \quad C^{\mathrm{co}}, \quad C^{\mathrm{t}}, \quad C^{\mathrm{to}} = (C^{\mathrm{o}})^{\mathrm{t}}, \quad C^{\mathrm{top}} = (C^{\mathrm{op}})^{\mathrm{t}}, \quad C^{\mathrm{cot}} = (C^{\mathrm{co}})^{\mathrm{t}}\,.$$

Note that this group of dualities is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, a natural basis (as a module over $\mathbb{Z}/2\mathbb{Z}$) being given by D_{op}, D_{co} and D_{t}.

We now recall the compatibilities of the dualities of $\omega\text{-}\mathcal{C}at$ with the Gray tensor product.

Proposition 2.23. — *Let A and B be two ω-categories. There are canonical isomorphisms*

$$(A \otimes B)^{\mathrm{op}} \simeq B^{\mathrm{op}} \otimes A^{\mathrm{op}}, \quad (A \otimes B)^{\mathrm{co}} \simeq B^{\mathrm{co}} \otimes A^{\mathrm{co}}, \quad (A \otimes B)^{\mathrm{o}} \simeq A^{\mathrm{o}} \otimes B^{\mathrm{o}}\,,$$

natural in A and B. In other words, the functors

$$D_{\mathrm{co}}, D_{\mathrm{op}} \colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$$

are anti-monoidal and the functor

$$D_{\mathrm{o}} \colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$$

is monoidal, ω-$\mathcal{C}at$ being endowed with the Gray tensor product.

Proof. — See for instance [**4**, Proposition A.22]. □

Remark 2.24. — Besides the trivial duality, D_{op}, D_{co} and D_{o} are the only dualities of ω-$\mathcal{C}at$ that are either monoidal or anti-monoidal (see the proof of [**4**, Proposition A.20]).

Remark 2.25. — It follows from the previous proposition that if A and B are two ω-categories, then we have natural isomorphisms

$$\begin{aligned} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)^{\mathrm{op}} &\simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(A^{\mathrm{op}},B^{\mathrm{op}})\,, \\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)^{\mathrm{co}} &\simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(A^{\mathrm{co}},B^{\mathrm{co}})\,, \\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)^{\mathrm{o}} &\simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A^{\mathrm{o}},B^{\mathrm{o}})\,. \end{aligned}$$

2.26. — Let $\mathbb{C}$ be a Gray ω-category. Using the previous proposition, from $\mathbb{C}$, we can get two anti Gray ω-categories $(D_{\mathrm{op}})_*(\mathbb{C})$ and $(D_{\mathrm{co}})_*(\mathbb{C})$, and one Gray ω-category $(D_{\mathrm{o}})_*(\mathbb{C})$, obtained by applying these dualities hom-wise (see 1.4 for the notation). Note that in these new Gray or anti Gray ω-categories, the 1-cells are never reversed and in some sense there is a shift of the reversed dimensions by 1. With this in mind, we set, with the notation of 1.2,

$$\mathbb{C}^{\mathrm{op}} = \big((D_{\mathrm{co}})_*(\mathbb{C})\big)^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{co}} = (D_{\mathrm{op}})_*(\mathbb{C}), \quad \mathbb{C}^{\mathrm{o}} = \big((D_{\mathrm{o}})_*(\mathbb{C})\big)^{\mathrm{t}}\,.$$

Then $\mathbb{C}^{\mathrm{op}}$ is a Gray ω-category, and $\mathbb{C}^{\mathrm{co}}$ and $\mathbb{C}^{\mathrm{o}}$ are anti Gray ω-categories. We also set

$$\mathbb{C}^{\mathrm{top}} = (\mathbb{C}^{\mathrm{op}})^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{cot}} = (\mathbb{C}^{\mathrm{co}})^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{to}} = (\mathbb{C}^{\mathrm{o}})^{\mathrm{t}}\,.$$

To sum up, from a Gray ω-category $\mathbb{C}$, we get three other Gray ω-categories

$$\mathbb{C}^{\mathrm{op}}, \quad \mathbb{C}^{\mathrm{cot}}, \quad \mathbb{C}^{\mathrm{to}}$$

and four anti Gray ω-categories

$$\mathbb{C}^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{top}}, \quad \mathbb{C}^{\mathrm{co}}, \quad \mathbb{C}^{\mathrm{o}}\,.$$

No other duality of ω-$\mathcal{C}at$ produces a Gray or an anti Gray ω-category.

2.27. — Using the previous paragraph, one can interpret the dualities D_{co}, D_{op} and D_{o} of ω-categories as Gray ω-functors. More precisely, one can check using 2.25 that these dualities induce isomorphisms of Gray ω-categories

$$\begin{aligned} D_{\mathrm{op}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{co}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,\\ D_{\mathrm{co}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{top}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,\\ D_{\mathrm{o}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,. \end{aligned}$$

Dually, we have isomorphisms of anti Gray ω-categories

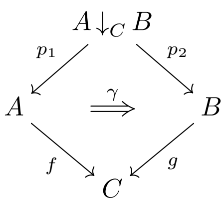
$$\begin{aligned} D'_{\mathrm{op}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{co}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,,\\ D'_{\mathrm{co}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{top}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,,\\ D'_{\mathrm{o}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{to}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,. \end{aligned}$$

3. Comma ω-categories

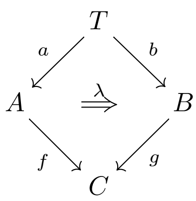
3.1. — Consider a diagram

$$A \xrightarrow{\;f\;} C \xleftarrow{\;g\;} B$$

in ω-$\mathcal{C}at$. The *comma ω-category* $A{\downarrow_C}B$, also denoted by $f{\downarrow}g$, is the universal ω-category endowed with a 2-square



in ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$, that is, where p_1 and p_2 are ω-functors and γ is an *oplax* transformation. This means that if T is an ω-category endowed with a similar 2-square



,

then there exists a unique ω-functor $h\colon T \to A{\downarrow_C}B$ that factors the diagram in the sense that

$$p_1 h = a, \quad p_2 h = b \quad \text{and} \quad \gamma *_0 h = \lambda\,.$$

By 2.8, and with its notation, the data of such a 2-square is equivalent to the data of an ω-functor from T to the limit of the diagram

$$\begin{array}{ccccccc} A & & & \Gamma C & & & B \\ & \searrow^{f} & \swarrow_{\mathbb{s}} & & \searrow^{\mathbb{t}} & \swarrow_{g} & \\ & C & & & & C & \end{array} \;.$$

This shows that we have

$$A \downarrow_C B = A \times_C \Gamma C \times_C B\,.$$

The canonical projections p_1 and p_2 are the obvious projections on A and B, and the 2-cell $\gamma\colon fp_1 \Rightarrow gp_2$ corresponds to the projection

$$\gamma\colon A \downarrow_C B \to \Gamma C\,.$$

Example 3.2. — The slice ω-categories are particular cases of comma ω-categories. Indeed, if C is an ω-category and c is an object of C, then the comma construction $c \downarrow C$ of the diagram

$$\mathrm{D}_0 \xrightarrow{\;c\;} C \xleftarrow{\;1_C\;} C$$

is canonically isomorphic to the slice ω-category ${}_{c}\backslash C$ described in [**4**, Chapter 9] (see [**3**, Proposition 7.1] for a proof). More generally, if $v\colon B \to C$ is an ω-functor, then we have

$${}_{c}\backslash B = c \downarrow v\,,$$

where ${}_{c}\backslash B$ is the relative slice defined by the pullback

$$\begin{array}{ccc} {}_{c}\backslash B & \longrightarrow & {}_{c}\backslash C \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle U} \\ B & \xrightarrow[v]{} & C \end{array} \;,$$

with U denoting the forgetful ω-functor.

Similarly, we have

$$C/_{c} = C \downarrow c$$

and, more generally, if $u\colon A \to C$ is an ω-functor,

$$A/_{c} = u \downarrow c\,.$$

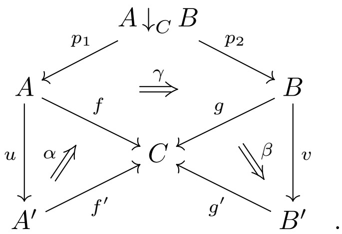
3.3. — The comma construction $A \downarrow_C B$ is functorial in A and B. Indeed, if

$$\begin{array}{ccccc} A & \xrightarrow{\;f\;} & & \xleftarrow{\;g\;} & B \\ {\scriptstyle u}\big\downarrow & \alpha \Nearrow & C & \Searrow \beta & \big\downarrow{\scriptstyle v} \\ A' & \xrightarrow[f']{} & & \xleftarrow[g']{} & B' \end{array}$$

is a diagram in ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$, then we get an ω-functor

$$(u, \alpha) \downarrow (\beta, v)\colon A \downarrow_C B \to A' \downarrow_C B'$$

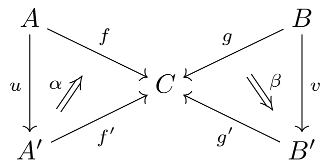
by applying the universal property of $A' \downarrow_C B'$ to the 2-square obtained by composing the diagram



Therefore, the comma construction defines a functor from the obvious category whose objects are the diagrams

$$A \xrightarrow{f} C \xleftarrow{g} B$$

and whose morphisms are the diagrams



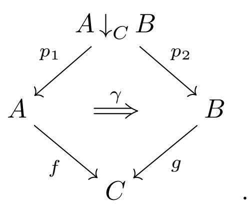
in ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ to the category ω-$\mathcal{C}at$.

In [**3**], the first-named author and Maltsiniotis proved that the comma construction can be promoted to a sesquifunctor. The main goal of the present paper is to express and prove the full functorialities of the comma construction, with respect to the higher structure of the Gray ω-category ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. To do so, one ingredient will be the ω-categorical universal property of the comma construction that we will now describe.

3.4. — Let

$$A \xrightarrow{f} C \xleftarrow{g} B$$

be as before and consider the universal 2-square



For every ω-category T, we have a canonical ω-functor

$$\begin{array}{c}\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, A\downarrow_C B)\\ \big\downarrow \\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,\Gamma C) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B)\end{array} ,$$

induced by the projections p_1, γ and p_2, which we can identify with an ω-functor

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, A\downarrow_C B) \longrightarrow \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A) \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B)$$

using the canonical isomorphism $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,\Gamma C) \simeq \Gamma\,\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)$ of 2.8. Since the functor $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,-) \colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$ is a right adjoint, it commutes with fiber products and we get the following result:

***Proposition 3.5* (Higher universal property of the comma construction)** *If $A \xrightarrow{f} C \xleftarrow{g} B$ is a diagram in ω-$\mathcal{C}at$, then for any ω-category T the canonical morphism*

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, A\downarrow_C B) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A) \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B)$$

is an isomorphism.

The enriched description of the ω-category of cylinders (see 2.12) leads to an analogous description for the comma ω-category:

3.6. — If $A \xrightarrow{f} C \xleftarrow{g} B$ is a diagram in ω-$\mathcal{C}at$, then the ω-category $A\downarrow_C B$ can be described (up to isomorphism) as a category enriched in ω-categories in the following way:

- The objects of $A\downarrow_C B$ are triples $(a, l\colon fa \to gb, b)$, where a is an object of A, b an object of B and l a 1-cell of C.
- If $(a, l\colon fa \to gb, b)$ and $(a', l'\colon fa' \to gb', b')$ are two objects of $A\downarrow_C B$, then
$$\underline{\mathsf{Hom}}_{A\downarrow_C B}((a,l,b),(a',l',b'))$$
$$= \underline{\mathsf{Hom}}_A(a,a') \times_{\underline{\mathsf{Hom}}_C(fa,gb')} \Gamma\underline{\mathsf{Hom}}_C(fa,gb') \times_{\underline{\mathsf{Hom}}_C(fa,gb')} \underline{\mathsf{Hom}}_B(b,b')\,,$$
where this iterated fiber product denotes the limit of the diagram
$$\begin{array}{ccccccc} \underline{\mathsf{Hom}}_A(a,a') & & & \Gamma\underline{\mathsf{Hom}}_C(fa,gb') & & & \underline{\mathsf{Hom}}_B(b,b') \\ & \searrow^{l' *_0 f(-)} & \swarrow_{\mathbb{s}} & & \searrow^{\mathbb{t}} & \swarrow_{g(-) *_0 l} & \\ & \underline{\mathsf{Hom}}_C(fa,gb') & & & & \underline{\mathsf{Hom}}_C(fa,gb') & \end{array} .$$
This ω-category is actually itself a comma ω-category, namely
$$\underline{\mathsf{Hom}}_A(a,a') \downarrow_{\underline{\mathsf{Hom}}_C(fa,gb')} \underline{\mathsf{Hom}}_B(b,b')\,.$$
Concretely, a k-cell in this hom is a triple (u,α,v) in
$$\underline{\mathsf{Hom}}_A(a,a')_k \times \Gamma\underline{\mathsf{Hom}}_C(fa,gb')_k \times \underline{\mathsf{Hom}}_B(b,b')_k$$

such that

$$\mathbb{s}(\alpha) = l' *_0 f(u) \quad \text{and} \quad \mathbb{t}(\alpha) = g(v) *_0 l\,.$$

– If $(a, l, b), (a', l', b'), (a'', l'', b'')$ are three objects of $A \downarrow_C B$, the composition ω-functor

$$\begin{gathered}\underline{\mathsf{Hom}}_{A\downarrow_C B}((a', l', b'), (a'', l'', b'')) \times \underline{\mathsf{Hom}}_{A\downarrow_C B}((a, l, b), (a', l', b')) \\ \downarrow \\ \underline{\mathsf{Hom}}_{A\downarrow_C B}((a, l, b), (a'', l'', b''))\end{gathered}$$

is given by

$$((u', \alpha', v'), (u, \alpha, v)) \mapsto (u' *_0 u, (g(v') *_l \alpha) *_c (\alpha' *_r f(u)), v' *_0 v)\,.$$

– If (a, l, b) is an object of $A \downarrow_C B$, its unit is the triple $(1_a, \mathbb{1}_l, 1_b)$.

3.7. — The comma construction we studied in this section is the *oplax* comma construction. Similarly, one can define a *lax comma construction* by replacing the oplax transformation in the 2-square of the universal property of the oplax comma construction by a lax transformation. If

$$A \xrightarrow{f} C \xleftarrow{g} B$$

is a diagram in ω-$\mathcal{C}at$, we will denote by $A \downarrow'_C B$, or $f \downarrow' g$, the lax comma construction of f and g. Explicitly, we have

$$A \downarrow'_C B = A \times_C \Gamma' C \times_C B\,,$$

where $\Gamma' C = \underline{\mathsf{Hom}}_{\text{oplax}}(\mathrm{D}_1, C)$.

The lax comma construction can also be defined by duality from the oplax version. Indeed, there are natural isomorphisms

$$(A \downarrow_C B)^{\text{op}} \simeq B^{\text{op}} \downarrow'_{C^{\text{op}}} A^{\text{op}} \quad \text{and} \quad (A \downarrow_C B)^{\text{co}} \simeq A^{\text{co}} \downarrow'_{C^{\text{co}}} B^{\text{co}}\,.$$

In particular, we have

$$(A \downarrow_C B)^{\text{o}} \simeq B^{\text{o}} \downarrow_{C^{\text{o}}} A^{\text{o}} \quad \text{and} \quad (A \downarrow'_C B)^{\text{o}} \simeq B^{\text{o}} \downarrow'_{C^{\text{o}}} A^{\text{o}}\,.$$

In this text, we will mainly deal with the oplax version of the comma construction and therefore drop the adjective "oplax".

4. Slices of Gray ω-categories

The purpose of this section is to define, for $\mathbb{C}$ a Gray ω-category and c an object of $\mathbb{C}$, a slice Gray ω-category $\mathbb{C}/_c$.

The description of the comma construction of 3.6 gives in particular an enriched description for the slices of ω-categories. We will see that this description can be adapted to Gray ω-categories. The definition will involve the ω-category

$$\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)) = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, \underline{\mathsf{Hom}}_{\mathbb{C}}(d,c))\,,$$

where d is another object of $\mathbb{C}$, and we start the section by an analysis of the structure of this ω-category.

4.1. — Let $\mathbb{C}$ be a Gray ω-category. For every object c of $\mathbb{C}$, by composing the two anti Gray ω-functors

$$\mathbb{C}^{\mathrm{t}} \xrightarrow{\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \xrightarrow{\quad\Gamma\quad} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,,$$

where

$$\Gamma = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, -)\,,$$

we get an anti Gray ω-functor

$$\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)) \colon \mathbb{C}^{\mathrm{t}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,.$$

By Proposition 1.10, this means that $\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c))$ is a right $\mathbb{C}$-module in the sense of 1.9.

4.2. — Let $\mathbb{C}$ be a Gray ω-category and let a, b and c be three objects of $\mathbb{C}$. The structure of right $\mathbb{C}$-module of the previous paragraph defines an ω-functor

$$*_r \colon \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \to \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\,.$$

Moreover, the axioms of modules give that, if a, b, c and d are four objects of $\mathbb{C}$, then the diagrams

$$\begin{array}{ccc}
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(c,d) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(c,d)\otimes *_0} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(c,d) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) \\
{\scriptstyle *_r \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\downarrow & & \downarrow{\scriptstyle *_r} \\
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,d) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow[*_r]{} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,d)
\end{array}$$

and

$$\begin{array}{ccc}
\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\otimes 1_a} & \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,a) \\
& {\searrow}{\scriptstyle =} & \downarrow{\scriptstyle *_r} \\
& & \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)
\end{array}$$

commute.

Remark 4.3. — In the case where the Gray ω-category comes from a strict ω-category, we have already defined an operation $*_r$ in 2.10. Proposition 4.15 will show that it is compatible with the operation $*_r$ introduced in the previous paragraph.

4.4. — We saw in 2.9 that, if C is an ω-category, then we have ω-functors

$$\mathbb{s}, \mathbb{t}\colon \Gamma C \to C, \quad \mathbb{k}\colon C \to \Gamma C \quad \text{and} \quad *_c\colon \Gamma C \times_C \Gamma C \to \Gamma C$$

that define a structure of category internal to ω-categories. All the operations of this structure are natural in C. This means that this structure of internal category to ω-$\mathcal{C}at$ extends to a structure of internal category to the category of anti Gray ω-functors from ω-$\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ to itself and strict transformations between them. More precisely, the anti Gray ω-functor $\Gamma\colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ is the object of morphisms of a category internal to the category of anti Gray ω-functors from ω-$\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ to itself, the object of objects being the identity anti Gray ω-functor.

4.5. — Let $\mathbb{C}$ be a Gray ω-category and let c be an object of $\mathbb{C}$. By precomposing the internal category of the previous paragraph with the anti Gray ω-functor

$$\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c)\colon \mathbb{C}^{\mathrm{t}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,,$$

we get that

$$\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c))\colon \mathbb{C}^{\mathrm{t}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$$

is the object of morphisms of a category internal to the category of anti Gray ω-functors from $\mathbb{C}^{\mathrm{t}}$ to ω-$\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, with object of objects $\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c)$ and structure maps

$$\mathbb{s}, \mathbb{t}\colon \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c) \Rightarrow \underline{\mathsf{Hom}}_{\mathbb{C}}(-, c)\,,\ \ \mathbb{k}\colon\ \underline{\mathsf{Hom}}_{\mathbb{C}}(-, c) \Rightarrow \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c)$$

and

$$*_c\colon \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c) \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)} \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c) \Rightarrow \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c)\,.$$

By Proposition 1.10, this means that $\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c))$ is the object of morphisms of a category internal to the category of right $\mathbb{C}$-modules and the four above structure maps correspond to morphisms of right $\mathbb{C}$-modules.

4.6. — Let $\mathbb{C}$ be a Gray ω-category and let a, b and c be three objects of $\mathbb{C}$. The fact that the structure maps $\mathbb{s}$, $\mathbb{t}$, $\mathbb{k}$ and $*_c$ of the previous paragraph correspond to morphisms of right $\mathbb{C}$-modules precisely means that the squares

$$\begin{array}{ccc}
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b) & \xrightarrow{*_r} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a, c) \\
{\scriptstyle \mathbb{e}\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\downarrow & & \downarrow{\scriptstyle \mathbb{e}} \\
\underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b) & \xrightarrow[*_0]{} & \underline{\mathsf{Hom}}_{\mathbb{C}}(a, c)
\end{array}\quad,$$

for $\mathbb{e}$ being $\mathbb{s}$ or $\mathbb{t}$,

$$\begin{array}{ccc}
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b) & \xrightarrow{*_r} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a, c) \\
{\scriptstyle \mathbb{k}\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\uparrow & & \uparrow{\scriptstyle \mathbb{k}} \\
\underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b) & \xrightarrow[*_0]{} & \underline{\mathsf{Hom}}_{\mathbb{C}}(a, c)
\end{array}$$

and

$$\begin{array}{ccc} \big(\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)}\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\big)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{*_c\otimes 1} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \\ \downarrow & & \downarrow{\scriptstyle *_r} \\ \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)}\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) & \xrightarrow[*_c]{} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) \end{array},$$

where the left vertical arrow is the composite

$$\begin{array}{c} \big(\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)}\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\big)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \\ {\scriptstyle \mathrm{can}}\downarrow \\ \big(\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\big)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\big(\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\big) \\ {\scriptstyle *_r\times_{*_0}*_r}\downarrow \\ \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)}\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) \end{array},$$

commute.

Remark 4.7. — Note that if a, b and c are three objects of a Gray ω-category $\mathbb{C}$, there is no natural ω-functor

$$\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\to\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\,,$$

and in particular, there is no natural structure of left $\mathbb{C}$-module on $\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,-)$. What is true is that $\Gamma'\underline{\mathsf{Hom}}_{\mathbb{C}}(a,-)$, where $\Gamma'(C)=\underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1,C)$, is naturally a left $\mathbb{C}$-module.

We can now define slice Gray ω-categories.

4.8. — Let $\mathbb{C}$ be a Gray ω-category and let c be an object of $\mathbb{C}$. We define the *slice Gray ω-category* $\mathbb{C}/_c$ in the following way:

– The objects of $\mathbb{C}/_c$ are pairs $(d, f\colon d\to c)$, where d is an object of $\mathbb{C}$ and f a 1-cell.

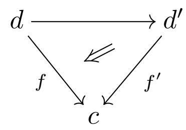
– If $(d, f\colon d\to c)$ and $(d', f'\colon d'\to c)$ are two objects of $\mathbb{C}$, we set

$$\begin{aligned}\underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d',f')) &= \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d')\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)}\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)}\{f\} \\ &= \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d')\downarrow_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)}\{f\}\,.\end{aligned}$$

By definition, a k-cell in this hom consists of a pair (u,α), with u a k-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}}(d,d')$ and α a k-cell of $\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)$ such that

$$\mathbb{s}(\alpha)=f'*_0 u \quad\text{and}\quad \mathbb{t}(\alpha)=1^k_f\,.$$

In particular, an object of this hom corresponds to a 2-triangle



We will denote by U and γ the projections

$$U\colon\ \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f')) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d')$$
$$\gamma\colon\ \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f')) \to \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)\,,$$

so that

$$U(u,\alpha) = u \quad\text{and}\quad \gamma(u,\alpha) = \alpha\,.$$

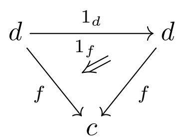
– If $(d, f\colon d \to c)$ is an object of $\mathbb{C}/c$, the associated unit

$$\mathrm{D}_0 \to \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d,f))$$

is given by the pair

$$\mathrm{D}_0 \xrightarrow{1_d} \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d)\,,\qquad \mathrm{D}_0 \xrightarrow{f} \underline{\mathsf{Hom}}_{\mathbb{C}}(d,c) \xrightarrow{\Bbbk} \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c))\,.$$

Concretely, it corresponds to the 2-triangle



In symbols, we have

$$1_{(d,f)} = (1_d, \mathbb{1}_f)$$

(remember that we denote $\Bbbk(f)$ by $\mathbb{1}_f$).

– Let (d,f), (d',f') and (d'',f'') be three objects of $\mathbb{C}/c$. We now define the composition ω-functor

$$\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f')) \to \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d'',f''))\,.$$

To define such an ω-functor we need to define two ω-functors

$$\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f')) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d'')$$
$$\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f')) \to \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c))$$

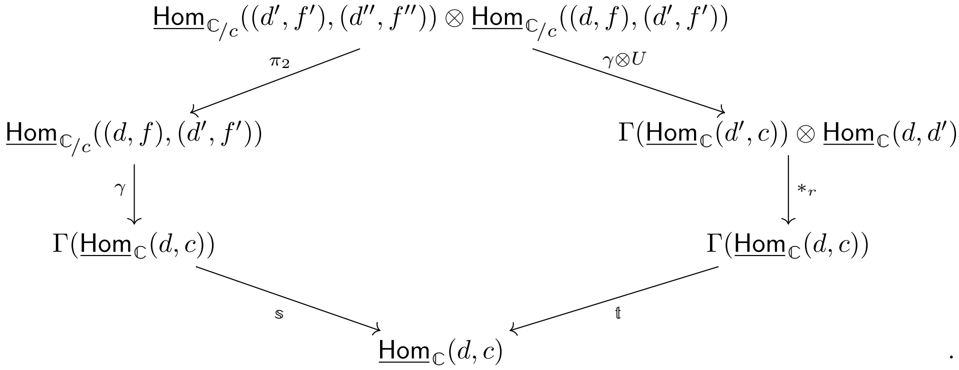
compatible with the pullback defining $\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d'',f''))$. The first one is defined by composing

$$\begin{array}{c}\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f')) \\ \Big\downarrow{\scriptstyle U\otimes U} \\ \underline{\mathsf{Hom}}_{\mathbb{C}}(d',d'') \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d') \xrightarrow{\quad *_0 \quad} \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d'')\end{array}\,.$$

The second one is defined by composing

$$\begin{array}{c} \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f')) \\ \downarrow \\ \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)) \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)) \xrightarrow{\ *_c\ } \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)) \end{array} \quad ,$$

where the vertical morphism is induced by the following hexagon

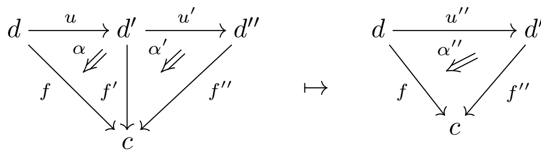


.

The fact that this hexagon commutes and that we have indeed defined a composition ω-functor will be verified within the proof of the next theorem. In symbols, we have

$$(u',\alpha') *_0 (u,\alpha) = \big(u' *_0 u, \alpha *_c (\alpha' *_r u)\big)\,.$$

Note that this is an ω-categorification of the composition of triangles



,

with

$$u'' = u'u \quad \text{and} \quad \alpha'' = \alpha *_1 (\alpha' *_0 u)\,.$$

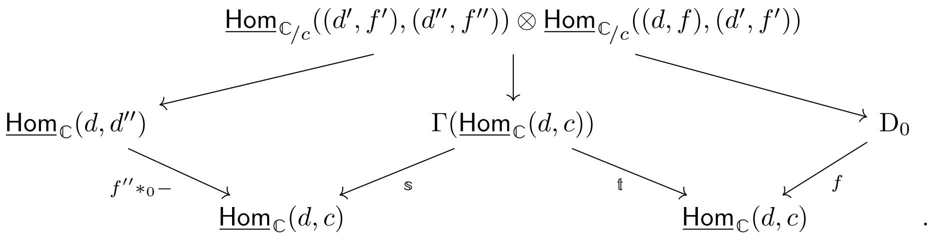
Theorem 4.9. — *If $\mathbb{C}$ is a Gray ω-category and c is an object of $\mathbb{C}$, then $\mathbb{C}/c$ as described above is indeed a Gray ω-category.*

Proof. — We start by proving that the composition of $\mathbb{C}/c$ described in the previous paragraph is well defined. We first have to prove that the hexagon claimed to be commutative is indeed commutative. Fix (u,α) a k-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d,f),(d',f'))$ and (u',α') a k-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d',f'),(d'',f''))$. If we evaluate the left part of the hexagon on $(u',\alpha')\otimes(u,\alpha)$, we get $\mathbb{s}(\alpha)$. Evaluating the right part, we get $\mathbb{t}(\alpha' *_r u)$. But

$$\mathbb{t}(\alpha' *_r u) = \mathbb{t}(\alpha') *_0 u = f' *_0 u = \mathbb{s}(\alpha)\,,$$

where the first equality follows from the fact that $\mathbb{t}$ is a morphism of right $\mathbb{C}$-modules (see the first square of 4.6 for $\mathbb{e} = \mathbb{t}$). A priori, we have shown that the hexagon commutes on "pure tensors" $(u', \alpha') \otimes (u, \alpha)$. Nevertheless, since each of the equalities we are using comes from commutative diagrams, this algebraic proof can be transformed into a diagrammatic proof showing that the hexagon commutes, without any restriction(2). From now on, we will freely use this technique to show commutativity of diagrams starting from a tensor product.

To prove that the composition of $\mathbb{C}/c$ is well defined, we now have to show the commutativity of the diagram



For the left square of the diagram, we have

$$\mathbb{s}(\alpha *_c (\alpha' *_r u)) = \mathbb{s}(\alpha' *_r u) = \mathbb{s}(\alpha') *_0 u = (f'' *_0 u') *_0 u = f'' *_0 (u' *_0 u)\,,$$

using first the internal category structure of the ω-category of cylinders (see 2.9), then the fact that $\mathbb{s}$ is a morphism of right $\mathbb{C}$-modules (see the first commutative square of 4.6 for $\mathbb{e} = \mathbb{s}$) and finally the associativity of the composition of the Gray ω-category $\mathbb{C}$. This proves that the left square commutes. As for the right square, we have

$$\mathbb{t}(\alpha *_c (\alpha' *_r u)) = \mathbb{t}(\alpha) = 1^k_f\,,$$

using again the internal category structure of the ω-category of cylinders. This ends the proof that the composition of $\mathbb{C}/c$ is well defined.

We now have to check the axioms of Gray ω-categories. Let us first prove the associativity. Fix a k-cell (u'', α'') of $\underline{\mathsf{Hom}}_{\mathbb{C}/c}((d'', f''), (d''', f'''))$. We have to prove that

$$\big(u'', \alpha''\big) *_0 \big((u', \alpha') *_0 (u, \alpha)\big) = \Big(u'' *_0 (u' *_0 u),\ \big(\alpha *_c (\alpha' *_r u)\big) *_c \big(\alpha'' *_r (u' *_0 u)\big)\Big)$$

equals

$$\big((u'', \alpha'') *_0 (u', \alpha')\big) *_0 \big(u, \alpha\big) = \Big((u'' *_0 u') *_0 u,\ \alpha *_c \big[\big(\alpha' *_c (\alpha'' *_r u')\big) *_r u\big]\Big)\,.$$

(2) A less elegant argument to conclude that the diagram commutes is to use the fact that the "pure tensors" $a \otimes b$ in a Gray tensor product $A \otimes B$ form a generating set under composition.

The equality of the first components follows from the associativity of the composition of $\mathbb{C}$. As for the second components, we have

$$\begin{aligned}\alpha *_c \big[\big(\alpha' *_c (\alpha'' *_r u')\big) *_r u\big] &= \alpha *_c \big[\big(\alpha' *_r u\big) *_c \big((\alpha'' *_r u') *_r u\big)\big] \\ &= \alpha *_c \big[\big(\alpha' *_r u\big) *_c \big(\alpha'' *_r (u' *_0 u)\big)\big] \\ &= \big(\alpha *_c (\alpha' *_r u)\big) *_c \big(\alpha'' *_r (u' *_0 u)\big)\,,\end{aligned}$$

where the first equality follows from the fact that $*_c$ is a morphism of right $\mathbb{C}$-modules (see the last commutative square of 4.6), the second from the fact that $*_r$ is a right action (see the commutative square of 4.2) and the last from the associativity of the operation $*_c$ (see 2.9). This ends the proof that the composition of $\mathbb{C}/_c$ is associative.

Finally, we prove the axioms involving units. For the right unit axiom, we have

$$(u,\alpha) *_0 1_{(d,f)} = (u,\alpha) *_0 (1_d, \mathbb{1}_f) = (u *_0 1_d, \mathbb{1}_f *_c (\alpha *_r 1_d)) = (u,\alpha)\,,$$

where the last equality uses the axiom of units in $\mathbb{C}$, the structure of category of 2.9 and the fact that $*_r$ is a right action (see the commutative triangle of 4.2). Finally, for the left unit axiom, we have

$$1_{(d',f')} *_0 (u,\alpha) = (1_{d'}, \mathbb{1}_{f'}) *_0 (u,\alpha) = (1_{d'} *_0 u, \alpha *_c (\mathbb{1}_{f'} *_r u)) = (u,\alpha)\,,$$

where the last equality uses the axiom of units in $\mathbb{C}$, the fact that $\mathbb{1}$ is a morphism of right $\mathbb{C}$-modules (see the second commutative square of 4.6) and the structure of internal category of 2.9. □

Remark 4.10. — The existence of slice Gray ω-categories was first conjectured by the first-named author and Maltsiniotis [**4**, conjecture C.24].

Remark 4.11. — The definition of the composition ω-functor of the slice Gray ω-category involves the right action $*_r$. We saw in 2.12 that for C a strict ω-category, we need both the right action $*_r$ and the left action $*_l$ to express the composition ω-functor of the ω-category of cylinders ΓC. But as noted in Remark 4.7, there is no left action $*_l$ for a Gray ω-category. This seems to indicate that, if $\mathbb{C}$ is a Gray ω-category, one cannot define a Gray ω-category of cylinders $\Gamma\mathbb{C}$. We will explain in Appendix A that this is indeed the case.

4.12. — Let $\mathbb{C}$ be a Gray ω-category and let c be an object of $\mathbb{C}$. We have a canonical Gray ω-functor

$$U\colon \mathbb{C}/_c \to \mathbb{C}\,,$$

called the *forgetful Gray ω-functor*. It is defined on objects by

$$(d,f) \mapsto d\,,$$

and, if $(d,f),(d',f')$ are two objects, on morphisms by the projection

$$U\colon\ \underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d',f')) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d')$$

(see 4.8).

We will now prove that our Gray slices are compatible with the slices of strict ω-categories. This requires comparing the operation $*_r$ in the Gray setting (see 4.2) with the one defined in the strict setting (see 2.10). To carry out this comparison, we introduce an alternative description of the operation $*_r$ in the Gray context.

4.13. — If C and D are two ω-categories, we will denote by

$$\nu\colon \Gamma C \otimes D \to \Gamma(C \otimes D)$$

the ω-functor obtained as the transpose of the ω-functor

$$\mathrm{D}_1 \otimes \Gamma C \otimes D \xrightarrow{\mathrm{ev}\otimes D} C \otimes D\,,$$

where ev is the evaluation ω-functor

$$\mathrm{D}_1 \otimes \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C) \to C\,.$$

Proposition 4.14. — *Let $\mathbb{C}$ be a Gray ω-category and let a, b and c be three objects of $\mathbb{C}$. Then the ω-functor $*_r$ of 4.2 can be described as the composite*

$$\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \xrightarrow{\nu} \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)) \xrightarrow{\Gamma(*_0)} \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\ .$$

Proof. — By adjunction, we have to show the commutativity of the square

$$\begin{array}{ccc}
\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{\nu^\sharp} & \underline{\mathsf{Hom}}_{\mathrm{lax}}(\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c), \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b))) \\
{\scriptstyle \underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)}\downarrow & & \downarrow{\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{lax}}(\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c),\Gamma(*_0))} \\
\underline{\mathsf{Hom}}_{\mathrm{lax}}(\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c), \underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)) & \xrightarrow[\Gamma]{} & \underline{\mathsf{Hom}}_{\mathrm{lax}}(\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c), \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c))
\end{array}\quad ,$$

where $\nu^\sharp$ denotes the transpose of ν. Note that the ω-functor $\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)$ is the transpose of the composition ω-functor $*_0\colon \underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)$. More generally, we claim that if X, Y and Z are three ω-categories and $m\colon X \otimes Y \to Z$ is any ω-functor, then the square

$$\begin{array}{ccc}
Y & \xrightarrow{\nu^\sharp} & \underline{\mathsf{Hom}}_{\mathrm{lax}}(\Gamma X, \Gamma(X \otimes Y)) \\
{\scriptstyle m^\sharp}\downarrow & & \downarrow{\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{lax}}(\Gamma X,\Gamma(m))} \\
\underline{\mathsf{Hom}}_{\mathrm{lax}}(X, Z) & \xrightarrow[\Gamma]{} & \underline{\mathsf{Hom}}_{\mathrm{lax}}(\Gamma X, \Gamma Z)
\end{array}\quad ,$$

where $m^\sharp$ denotes the transpose of m, commutes. By adjunction, this comes down to the commutativity of the square

$$\begin{array}{ccccc}
\mathrm{D}_1 \otimes \Gamma X \otimes Y & & \xrightarrow{\mathrm{ev}\otimes Y} & & X \otimes Y \\
{\scriptstyle \mathrm{D}_1\otimes\Gamma X\otimes m^\sharp}\downarrow & & & & \downarrow{\scriptstyle m} \\
\mathrm{D}_1 \otimes \Gamma X \otimes \underline{\mathsf{Hom}}_{\mathrm{lax}}(X, Z) & \xrightarrow[\mathrm{ev}\otimes\underline{\mathsf{Hom}}_{\mathrm{lax}}(X,Z)]{} & X \otimes \underline{\mathsf{Hom}}_{\mathrm{lax}}(X, Z) & \xrightarrow[\mathrm{ev}]{} & Z
\end{array}\quad ,$$

where ev denotes the evaluation ω-functor, which is readily checked. ☐

Proposition 4.15. — *Let C be an ω-category and let a, b and c be three objects of C. Then the triangle*

$$\begin{array}{ccc} \Gamma\,\underline{\mathsf{Hom}}_C(b,c)\otimes\underline{\mathsf{Hom}}_C(a,b) & \xrightarrow{*_r} & \Gamma\,\underline{\mathsf{Hom}}(a,c) \\ {\scriptstyle\pi}\big\downarrow & \nearrow{\scriptstyle *_r} & \\ \Gamma\,\underline{\mathsf{Hom}}_C(b,c)\times\underline{\mathsf{Hom}}_C(a,b) & & \end{array},$$

where the horizontal arrow is the ω-functor of 4.2 for the Gray ω-category associated to C and the diagonal arrow is the ω-functor of 2.10, commutes.

Proof. — Using the previous proposition, this boils down to showing the commutativity of the diagram

$$\begin{array}{ccccc} \Gamma\,\underline{\mathsf{Hom}}_C(b,c)\otimes\underline{\mathsf{Hom}}_C(a,b) & \xrightarrow{\nu} & \Gamma(\underline{\mathsf{Hom}}_C(b,c)\otimes\underline{\mathsf{Hom}}_C(a,b)) & \xrightarrow{\Gamma(*_0)} & \Gamma\,\underline{\mathsf{Hom}}_C(a,c) \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\Gamma(\pi)}\big\downarrow & \nearrow{\scriptstyle\Gamma(*_0)} & \\ \Gamma\,\underline{\mathsf{Hom}}_C(b,c)\times\underline{\mathsf{Hom}}_C(a,b) & \xrightarrow[1\times\Bbbk]{} & \Gamma(\underline{\mathsf{Hom}}_C(b,c)\times\underline{\mathsf{Hom}}_C(a,b)) & & \end{array},$$

where the definition of the bottom-horizontal arrow uses the identification

$$\Gamma(\underline{\mathsf{Hom}}_C(b,c)\times\underline{\mathsf{Hom}}_C(a,b))\simeq\Gamma\,\underline{\mathsf{Hom}}_C(b,c)\times\Gamma\,\underline{\mathsf{Hom}}_C(a,b)\,.$$

The triangle of the diagram obviously commutes and it suffices to show that the square commutes. More generally, we claim that if X and Y are two ω-categories, then the diagrams

$$\begin{array}{ccc} \Gamma X\otimes Y & \xrightarrow{\nu} & \Gamma(X\otimes Y) \\ & \searrow{\scriptstyle\pi_1} & \big\downarrow{\scriptstyle\Gamma(\pi_1)} \\ & & \Gamma X \end{array} \qquad\text{and}\qquad \begin{array}{ccc} \Gamma X\otimes Y & \xrightarrow{\nu} & \Gamma(X\otimes Y) \\ {\scriptstyle\pi_2}\big\downarrow & & \big\downarrow{\scriptstyle\Gamma(\pi_2)} \\ Y & \xrightarrow[\Bbbk]{} & \Gamma Y \end{array}$$

commute. This follows from the naturality of ν applied to the ω-functors $Y\to\mathrm{D}_0$ and $X\to\mathrm{D}_0$, respectively. ☐

Proposition 4.16. — *Let C be an ω-category and let c be an object of C. Then we have a canonical natural isomorphism*

$$\iota(C)/_c\simeq\iota(C/_c)\,,$$

commuting with the forgetful morphisms, where ι denotes the inclusion functor from ω-categories to Gray ω-categories.

Proof. — This is true by design of slice Gray ω-categories, and more precisely, by the description of slice ω-categories that follows from the description of comma ω-categories given in 3.6 and Proposition 4.15. ☐

Proposition 4.17. — *Let $\mathbb{C}$ be a Gray ω-category and let c and d be two objects of $\mathbb{C}$. Then the fiber of the forgetful Gray ω-functor $\mathbb{C}/_{c} \to \mathbb{C}$ at d is canonically isomorphic to $\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)^{\mathrm{o}}$.*

Proof. — Denote by U_d this fiber. By definition, its objects are 1-cells $d \to c$ of $\mathbb{C}$, and if f and f' are two such objects, we have

$$\underline{\mathsf{Hom}}_{U_d}(f,f') = \{f'\} \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c) \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \{f\} = \{f'\}\downarrow_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)}\{f\}\,.$$

This means that a k-cell of this ω-category is a k-cylinder α in $\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)$ such that $\mathbb{s}(\alpha) = 1_{f'}$ and $\mathbb{t}(\alpha) = 1_f$. Moreover, the composition ω-functor simplifies to

$$\begin{array}{ccc} \underline{\mathsf{Hom}}_{U_d}(f',f'') \otimes \underline{\mathsf{Hom}}_{U_d}(f,f') & \to & \underline{\mathsf{Hom}}_{U_d}(f,f'') \\ (\alpha',\alpha) & \mapsto & \alpha' *_c \alpha \end{array}$$

(as $\alpha' *_c (\alpha *_r 1_d) = \alpha' *_c \alpha$).

But in general, if a and a' are two objects of an ω-category A, we have a canonical isomorphism

$$\{a\}\downarrow_A\{a'\} \xrightarrow{\sim} \underline{\mathsf{Hom}}_A(a,a')^{\mathrm{o}}$$

(see [**1**, Proposition B.6.2]), sending a k-cylinder α in A to its principal cell α_k (see 2.8). We thus have

$$\underline{\mathsf{Hom}}_{U_d}(f,f') \simeq \underline{\mathsf{Hom}}_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)}(f',f)^{\mathrm{o}} = \underline{\mathsf{Hom}}_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)^{\mathrm{o}}}(f,f')\,,$$

this isomorphism sending a k-cylinder α in $\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)$ to its principal cell α_k. Now if (α',α) is in $\underline{\mathsf{Hom}}_{U_d}(f',f'')_k \times \underline{\mathsf{Hom}}_{U_d}(f,f')_k$, then

$$(\alpha' *_c \alpha)_k = \alpha'_k *_0 \alpha_k\,.$$

This shows that the isomorphism

$$\underline{\mathsf{Hom}}_{U_d}(f,f') \simeq \underline{\mathsf{Hom}}_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)^{\mathrm{o}}}(f,f')$$

is compatible with compositions, thereby proving the result. □

4.18. — Let C be a strict ω-category and let c be an object of C. If D is any duality of ω-$\mathcal{C}at$ (see 2.22), then by "conjugating" the slice construction $C/_{c}$ by D, one gets another slice construction. If D does not reverse the 1-cells, we set

$$C\overset{\mathrm{D}}{/}c = D\big(D(C)/c\big)\,.$$

In the case where D reverses the 1-cells, this notation would be misleading as $D\big(D(C)/_c\big)$ is actually a slice below c (and not above c). In particular, if D is the total dual, following the notation of [**4**], we set

$$c\backslash C = (C^{\mathrm{o}}/c)^{\mathrm{o}}\,.$$

Now if D is a general duality reversing the 1-cells, then denoting by D' the unique duality such that $D' \circ D$ is the total dual (in particular, D' does not reverse the 1-cells), one gets

$$D\big(D(C)/c\big) = D'\Big(\big(D'(C)^{\mathrm{o}}/c\big)^{\mathrm{o}}\Big) = D'\big(c\backslash D'(C)\big)\,.$$

Therefore, if D reverses the 1-cells, we set

$$c\overset{\mathrm{D}'}{\backslash}C = D\big(D(C)/c\big)\,.$$

This means that we only decorate (over or under) slices by dualities that do not reverse the 1-cells.

Let us now apply this to Gray ω-categories. If $\mathbb{C}$ is a Gray ω-category and c is an object of $\mathbb{C}$, we can conjugate our Gray slice construction by the three non-trivial dualities of Gray ω-categories (see 2.26) and we set

$$c\overset{\mathrm{co}}{\backslash}\mathbb{C} = (\mathbb{C}^{\mathrm{op}}/c)^{\mathrm{op}}\,, \qquad c\overset{\mathrm{top}}{\backslash}\mathbb{C} = (\mathbb{C}^{\mathrm{cot}}/c)^{\mathrm{cot}}\,, \qquad \mathbb{C}\overset{\mathrm{to}}{/}c = (\mathbb{C}^{\mathrm{to}}/c)^{\mathrm{to}}\,.$$

Note that each of these Gray ω-categories admits a forgetful Gray ω-functor to $\mathbb{C}$.

Similarly, if $\mathbb{C}$ is an anti Gray ω-category and c is an object of $\mathbb{C}$, we set

$$c\backslash\mathbb{C} = (\mathbb{C}^{\mathrm{o}}/c)^{\mathrm{o}}\,, \quad \mathbb{C}\overset{\mathrm{co}}{/}c = (c\backslash\mathbb{C}^{\mathrm{op}})^{\mathrm{op}}\,, \quad \mathbb{C}\overset{\mathrm{top}}{/}c = (c\backslash\mathbb{C}^{\mathrm{cot}})^{\mathrm{cot}}\,, \quad c\overset{\mathrm{to}}{\backslash}\mathbb{C} = (c\backslash\mathbb{C}^{\mathrm{to}})^{\mathrm{to}}\,.$$

Each of these anti Gray ω-categories admits a forgetful anti Gray ω-functor to $\mathbb{C}$.

5. Gray functorialities of the comma construction

The purpose of this section is to extend the comma construction

$$A \longrightarrow C \longleftarrow B \qquad \mapsto \qquad A{\downarrow_C}B$$

to a Gray ω-functor

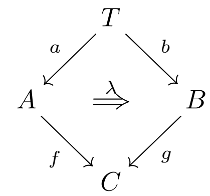
$$-{\downarrow_C}-\colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,$$

where $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$ is the slice Gray ω-category obtained by applying Theorem 4.9 to the Gray ω-category $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$, the slice $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C$ is obtained by duality (see 4.18) and the product is the categorical product of Gray ω-categories (see 1.3).

5.1. — Let us fix ω-functors

$$A \xrightarrow{\;f\;} C \xleftarrow{\;g\;} B \qquad \text{and} \qquad A' \xrightarrow{\;f'\;} C \xleftarrow{\;g'\;} B'\ .$$

To any 2-square



in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$, we are going to associate an ω-functor

$$K\colon \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}/C}}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}^{\mathrm{to}}_{\mathrm{oplax}/C}}((B,g),(B',g'))$$
$$\big\downarrow$$
$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B') \quad .$$

By definition, we have

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}/C}}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,A') \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)} \Gamma\, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)} \{f\}\end{aligned}$$

and, as the duality D_{to} consists in applying the total dual hom-wise, we get

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}^{\mathrm{to}}_{\mathrm{oplax}/C}}((B,g),(B',g'))\\ &\quad = \Big(\underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,B')^{\mathrm{o}} \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)^{\mathrm{o}}} \Gamma\big(\underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)^{\mathrm{o}}\big) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)^{\mathrm{o}}} \{g\}\Big)^{\mathrm{o}}\\ &\quad \simeq \{g\} \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)} \Gamma\, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,B')\,,\end{aligned}$$

as

$$\big(\Gamma(X^{\mathrm{o}})\big)^{\mathrm{o}} \simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1^{\mathrm{o}}, X)\,,$$

which is isomorphic to ΓX but in a way that exchanges $\mathbb{s}$ and $\mathbb{t}$.

This means that a k-cell in

$$\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}/C}}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}^{\mathrm{to}}_{\mathrm{oplax}/C}}((B,g),(B',g'))$$

consists of a 4-tuple

$$(u,\alpha,\beta,v)$$

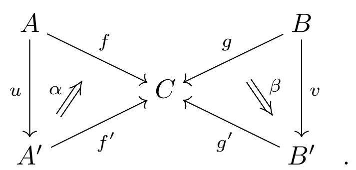
in

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,A')_k \times (\Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C))_k \times (\Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C))_k \times \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,B')_k$$

satisfying

$$\mathbb{s}(\alpha) = f' \ast_0 u, \quad \mathbb{t}(\alpha) = f, \quad \mathbb{s}(\beta) = g, \quad \mathbb{t}(\beta) = g' \ast_0 v\,.$$

In particular, for $k = 0$, we get a diagram



Similarly, a k-cell of

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B')$$

consists of a triple

$$(a',\lambda',b')$$

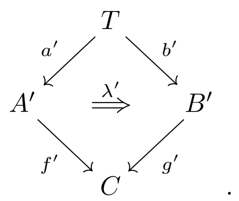
in

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, A')_k \times (\Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, C))_k \times \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, B')_k$$

such that

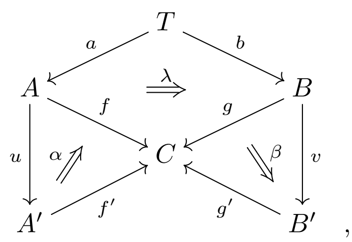
$$\mathbb{s}(\lambda') = f' *_0 a' \quad \text{and} \quad \mathbb{t}(\lambda') = g' *_0 b' \,.$$

For $k = 0$, we get a 2-square



.

The ω-functor K is defined by ω-categorification of the formula giving the total composite of the 2-diagram



,

that is, by the formula

$$(u, \alpha, \beta, v) \mapsto (u *_0 a, (\beta *_r b) *_c \lambda *_c (\alpha *_r a), v *_0 b)\,,$$

where $*_c$ denotes the internal composition of cylinders of 2.9 and $*_r$ the right action of 4.2. This formula is well defined as, first, by 4.2,

$$\mathbb{s}(\beta *_r b) = \mathbb{s}(\beta) *_0 b = g *_0 b = \mathbb{t}(\lambda) \quad \text{and} \quad \mathbb{t}(\alpha *_r a) = \mathbb{t}(\alpha) *_0 a = f *_0 a = \mathbb{s}(\lambda)\,,$$

so that $\lambda' = (\beta *_r b) *_c \lambda *_c (\alpha *_r a)$ makes sense, and, second, using 2.9 and again 4.2,

$$\mathbb{s}(\lambda') = \mathbb{s}(\alpha *_r a) = \mathbb{s}(\alpha) *_0 a = (f' *_0 u) *_0 a = f' *_0 (u *_0 a)\,,$$

and similarly

$$\mathbb{t}(\lambda') = g' *_0 (v *_0 b)\,.$$

Moreover, this formula is ω-functorial in (u, α, β, v) as it only involves the operations $*_0$, $*_c$ and $*_r$, which are all ω-functorial. (We could also have defined the ω-functor K in a more categorical way, as we did for the composition $*_0$ of Gray slices in 4.8.)

5.2. — If we apply the construction of the previous paragraph to the universal 2-square

$$\begin{array}{ccccc} & & A\downarrow_C B & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ A & & \overset{\gamma}{\Longrightarrow} & & B \\ & \searrow_{f} & & \swarrow_{g} & \\ & & C & & \end{array} \quad ,$$

we get an ω-functor

$$\begin{gathered} -\downarrow_C - \colon \underline{\mathsf{Hom}}_{\omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}/C}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C}((B,g),(B',g')) \\ \Big\downarrow \\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A\downarrow_C B, A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A\downarrow_C B, C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A\downarrow_C B, B') \\ \simeq \\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A\downarrow_C B, A'\downarrow_C B') \quad , \end{gathered}$$

where the isomorphism is the higher universal property of the comma construction (see Proposition 3.5). Explicitly, the ω-functor $-\downarrow_C -$ is given by the formula

$$(u,\alpha,\beta,v) \mapsto (u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)\,.$$

In particular, if $g\colon B \to C$ is an ω-functor, we have

$$\begin{aligned} (-\downarrow_C B)(u,\alpha) &= (-\downarrow_C -)(u,\alpha,\mathbb{1}_g,1_B) \\ &= (u *_0 p_1, (\mathbb{1}_g *_r p_2) *_c \gamma *_c (\alpha *_r p_1), 1_B *_0 p_2) \\ &= (u *_0 p_1, \gamma *_c (\alpha *_r p_1), p_2)\,, \end{aligned}$$

where the last equality uses the fact that $\mathbb{1}$ is a morphism of right $\omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}$-modules (see the second commutative square of 4.6) and the structure of internal category of 2.9, and, similarly, if $f\colon A \to C$ is an ω-functor, we have

$$(A\downarrow_C -)(\beta,v) = (p_1, (\beta *_r p_2) *_c \gamma, v *_0 p_2)\,.$$

Theorem 5.3. — *Let C be an ω-category. The comma construction extends, via the construction of the previous paragraph, to a Gray ω-functor*

$$-\downarrow_C - \colon \omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}/C \times \omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}\,.$$

Proof. — In this proof, to make our formula more compact, we set

$$\underline{\mathsf{Hom}}_{\mathrm{ol}} = \underline{\mathsf{Hom}}_{\mathrm{oplax}}\ , \quad \underline{\mathsf{Hom}}_{/C} = \underline{\mathsf{Hom}}_{\omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}/C} \quad \text{and} \quad \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/}C} = \underline{\mathsf{Hom}}_{\omega\text{-}\mathsf{Cat}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C}\,.$$

For the same reason, if $(T, T \to C)$ is an ω-category over C, we will denote it simply by T.

Let us now prove the result. We have to check the compatibility with the unit and the compatibility with the composition. To do so, we will use the same technique as in the proof of Theorem 4.9.

For the unit, we have

$$\begin{aligned}(-\downarrow_C -)\big(1_{(A,f)}, 1_{(B,g)}\big) &= (-\downarrow_C -)(1_A, \mathbb{1}_f, \mathbb{1}_g, 1_B)\\ &= (1_A *_0 p_1, (\mathbb{1}_g *_r p_2) *_c \gamma *_c (\mathbb{1}_f *_r p_1), 1_B *_0 p_2)\\ &= (p_1, \gamma, p_2)\\ &= 1_{A\downarrow_C B}\,,\end{aligned}$$

using the fact that $\mathbb{1}$ is a morphism of right ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$-modules (see the second commutative square of 4.6) and the structure of internal category of 2.9.

Let us now check the compatibility with the composition. Consider A, A', A'', B, B', B'' six ω-categories over C. We have to prove that the two canonical ω-functors

$$\begin{gathered}\big(\underline{\mathsf{Hom}}_{/C}(A', A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B', B'')\big) \otimes \big(\underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B')\big)\\ \big\downarrow\\ \underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B, A''\downarrow_C B'')\quad ,\end{gathered}$$

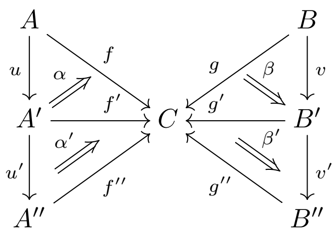
which we will describe below, are equal. Consider

$$(u', \alpha', \beta', v') \quad \text{and} \quad (u, \alpha, \beta, v)$$

cells of

$$\underline{\mathsf{Hom}}_{/C}(A', A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B', B'') \quad \text{and} \quad \underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B')$$

respectively. When these cells are 0-cells, we get a diagram



.

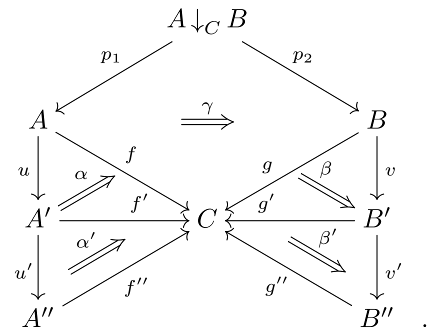
Consider first the ω-functor M defined as the composite

$$\begin{gathered}\big(\underline{\mathsf{Hom}}_{/C}(A', A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B', B'')\big) \otimes \big(\underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B')\big)\\ \mathrm{can}\big\downarrow\\ \big(\underline{\mathsf{Hom}}_{/C}(A', A'') \otimes \underline{\mathsf{Hom}}_{/C}(A, A')\big) \times \big(\underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B', B'') \otimes \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B')\big)\\ *_0 \otimes *_0\big\downarrow\\ \underline{\mathsf{Hom}}_{/C}(A, A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B'')\\ -\downarrow_C -\big\downarrow\\ \underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B, A''\downarrow_C B'')\quad .\end{gathered}$$

We have

$$\begin{aligned}
&M\big((u',\alpha',\beta',v')\otimes(u,\alpha,\beta,v)\big)\\
&\quad= \big(-\downarrow_C -\big)\big((u',\alpha') *_0 (u,\alpha), (\beta',v') *_0 (\beta,v)\big)\\
&\quad= \big(-\downarrow_C -\big)\big(u' *_0 u, \alpha *_c (\alpha' *_r u), (\beta' *_r v) *_c \beta, v' *_0 v\big)\\
&\quad= \Big((u' *_0 u) *_0 p_1,\\
&\qquad\quad \big[((\beta' *_r v) *_c \beta) *_r p_2\big] *_c \gamma *_c \big[(\alpha *_c (\alpha' *_r u)) *_r p_1\big],\\
&\qquad\quad (v' *_0 v) *_0 p_2\Big)\,.
\end{aligned}$$

Note that this last formula is an ω-categorification of the formula for the total composite of the 2-diagram

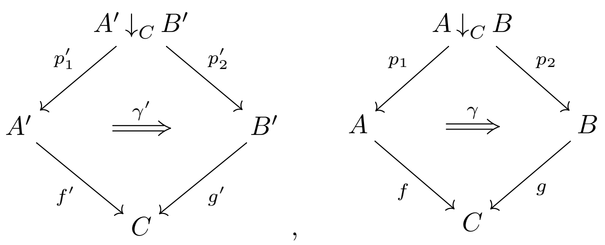


.

Consider now the ω-functor N obtained as the composite

$$\begin{gathered}
\big(\underline{\mathsf{Hom}}_{/C}(A',A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B',B'')\big) \otimes \big(\underline{\mathsf{Hom}}_{/C}(A,A') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B,B')\big)\\
\Big\downarrow {\scriptstyle (-\downarrow_C -)\otimes(-\downarrow_C -)}\\
\underline{\mathsf{Hom}}_{\mathrm{ol}}(A'\downarrow_C B', A''\downarrow_C B'') \otimes \underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B, A'\downarrow_C B')\\
\Big\downarrow {\scriptstyle \circ}\\
\underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B, A''\downarrow_C B'')\quad .
\end{gathered}$$

Let us compute $N\big((u',\alpha',\beta',v')\otimes(u,\alpha,\beta,v)\big)$. Denote by



the two universal 2-squares involved. By definition, we have

$$(-\downarrow_C -)(u',\alpha',\beta',v') = (u' *_0 p'_1, (\beta' *_r p'_2) *_c \gamma' *_c (\alpha' *_r p'_1), v' *_0 p'_2)\,,$$
$$(-\downarrow_C -)(u,\alpha,\beta,v) = (u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)\,,$$

and we have to compute the composition of these two cells in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. We have

$$\begin{aligned}
&N\big((u',\alpha',\beta',v') \otimes (u,\alpha,\beta,v)\big)\\
&\quad= \Big(u' *_0 p'_1, (\beta' *_r p'_2) *_c \gamma' *_c (\alpha' *_r p'_1), v' *_0 p'_2\Big)\\
&\qquad\quad *_0 \Big(u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2\Big)\\
&\quad= \Big(u' *_0 (u *_0 p_1),\\
&\qquad\quad \big[\beta' *_r (v *_0 p_2)\big] *_c \big[(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)\big] *_c \big[\alpha' *_r (u *_0 p_1)\big],\\
&\qquad\quad v' *_0 (v *_0 p_2)\Big)\,.
\end{aligned}$$

The axiom we are checking is thus equivalent to the equalities

$$u' *_0 (u *_0 p_1) = (u' *_0 u) *_0 p_1\,, \qquad v' *_0 (v *_0 p_2) = (v' *_0 v) *_0 p_2$$

and

$$\begin{aligned}
&\big[\beta' *_r (v *_0 p_2)\big] *_c \big[(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)\big] *_c \big[\alpha' *_r (u *_0 p_1)\big]\\
&\qquad= \big[((\beta' *_r v) *_c \beta) *_r p_2\big] *_c \gamma *_c \big[(\alpha *_c (\alpha' *_r u)) *_r p_1\big]\,.
\end{aligned}$$

The two first equalities are obviously true. As for the last one, we have

$$\begin{aligned}
&\big[\beta' *_r (v *_0 p_2)\big] *_c \big[(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)\big] *_c \big[\alpha' *_r (u *_0 p_1)\big]\\
&\quad= \Big(\big[\beta' *_r (v *_0 p_2)\big] *_c \big[\beta *_r p_2\big]\Big) *_c \gamma *_c \Big(\big[\alpha *_r p_1\big] *_c \big[\alpha' *_r (u *_0 p_1)\big]\Big)\\
&\quad= \Big(\big[(\beta' *_r v) *_r p_2\big] *_c \big[\beta *_r p_2\big]\Big) *_c \gamma *_c \Big(\big[\alpha *_r p_1\big] *_c \big[(\alpha' *_r u) *_r p_1\big]\Big)\\
&\quad= \big[((\beta' *_r v) *_c \beta) *_r p_2\big] *_c \gamma *_c \big[(\alpha *_c (\alpha' *_r u)) *_r p_1\big]\,,
\end{aligned}$$

where the first equality follows from the associativity of $*_c$, the second from the fact that $*_r$ is a right module action (see 4.2) and the last one from the fact that this action is compatible with $*_c$ (see the last square of 4.6). □

Corollary 5.4. — *If $B \to C$ is an ω-functor, then we have a Gray ω-functor*

$$-\downarrow_C B \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,$$

and if $A \to C$ is an ω-functor, then we have a Gray ω-functor

$$A\downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

Remark 5.5. — If A and B are two fixed ω-categories, there is a canonical embedding

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)^{\mathrm{o}} \times \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C\,.$$

Indeed, by Proposition 4.17, the ω-category $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)^{\mathrm{o}}$ canonically embeds in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$, and, by duality, this implies that

$$\left(\underline{\mathsf{Hom}}_{(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}}}(B,C)^{\mathrm{o}}\right)^{\mathrm{to}} = \underline{\mathsf{Hom}}_{(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}}}(B,C)^{\mathrm{t}} = \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}}(B,C)^{\mathrm{to}}$$

embeds in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C$.

The comma construction thus restricts to a Gray ω-functor

$$-\downarrow_C -\colon \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)^{\mathrm{o}} \times \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)^{\mathrm{to}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

Remark 5.6. — By duality, one can deduce that the *lax* comma construction $-\downarrow'-$ (see 3.7) defines an *anti* Gray ω-functor. This follows from the formula $A\downarrow'_C B \simeq \left(A^{\mathrm{co}}\downarrow_{C^{\mathrm{co}}} B^{\mathrm{co}}\right)^{\mathrm{co}}$ (see again 3.7). Indeed, by 2.27, the operation $X \mapsto X^{\mathrm{co}}$ defines an isomorphism of Gray ω-categories $D_{\mathrm{co}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{top}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ and an isomorphism of anti Gray ω-categories $D'_{\mathrm{co}}\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{top}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, and we can consider the chain of anti Gray ω-functors

$$\begin{array}{c}
\left((\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{top}}/C\right)^{\mathrm{top}} \times \left((\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{top}}\overset{\mathrm{to}}{/}C\right)^{\mathrm{top}} \\
\big\downarrow \\
\left(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C^{\mathrm{co}}\right)^{\mathrm{top}} \times \left(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C^{\mathrm{co}}\right)^{\mathrm{top}} \xrightarrow{-\downarrow_{C^{\mathrm{co}}}-} (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{top}} \xrightarrow{D'_{\mathrm{co}}} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}
\end{array}\quad,$$

where the vertical arrow is induced by D_{co}. Composing this chain, we get an anti Gray ω-functor

$$-\downarrow'_C -\colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\overset{\mathrm{top}}{/}C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\overset{\mathrm{co}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,.$$

Remark 5.7. — The two Gray ω-functors of Corollary 5.4 can be deduced from each other using the formula $A\downarrow_C B \simeq \left(B^{\mathrm{o}}\downarrow_{C^{\mathrm{o}}} A^{\mathrm{o}}\right)^{\mathrm{o}}$ of 3.7. For instance, if $u\colon A \to C$ is an ω-functor, the Gray ω-functor $A\downarrow_C -$ can be identified with the composite of the Gray ω-functors

$$\left((\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}}/C\right)^{\mathrm{to}} \xrightarrow{D_{\mathrm{o}}} \left(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C^{\mathrm{o}}\right)^{\mathrm{to}} \xrightarrow{-\downarrow_{C^{\mathrm{o}}}A^{\mathrm{o}}} (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}} \xrightarrow{D_{\mathrm{o}}} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

We end the section by expressing that the comma construction of $A\downarrow_C B$ is above A and B via a strict transformation.

Proposition 5.8. — *Let C be an ω-category. The canonical projection*

$$p = (p_1, p_2)\colon A\downarrow_C B \to A \times B$$

is natural in

$$A \longrightarrow C \longleftarrow B\,,$$

in the sense that it defines a strict transformation (see 2.14) from the Gray ω-functor

$$-\downarrow_C -\colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$$

to the Gray ω-functor obtained by composing

$$\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C \xrightarrow{U\times U} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \xrightarrow{\times} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,$$

where U denotes the forgetful Gray ω-functor (see 4.12) and $\times$ is the product Gray ω-functor (see 1.6).

Proof. — Let A, A', B and B' be four ω-categories above C. Using the same abbreviation as in the proof of Theorem 5.3, we have to show the commutativity of the square

$$\begin{array}{ccc} \underline{\mathsf{Hom}}_{/C}(A,A') \times \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/}C}(B,B') & \xrightarrow{-\downarrow_C -} & \underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B, A'\downarrow_C B') \\ {\scriptstyle\times}\big\downarrow & & \big\downarrow{\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B,p')} \\ \underline{\mathsf{Hom}}_{\mathrm{ol}}(A\times B, A'\times B') & \xrightarrow[\underline{\mathsf{Hom}}_{\mathrm{ol}}(p,A'\times B')]{} & \underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B, A'\times B') \end{array}\quad,$$

where we denoted simply by $\times$ the target Gray ω-functor of the statement. So let (u,α,β,v) be a cell in the source ω-category of this square. Using the formula defining the Gray comma construction, we get that this cell is sent to $(u *_0 p_1, v *_0 p_2)$ in $\underline{\mathsf{Hom}}_{\mathrm{ol}}(A\downarrow_C B, A'\times B')$ by the upper path of the square. But the Gray ω-functor $\times$ sends this same cell to $(u *_0 q_1, v *_0 q_2)$ in $\underline{\mathsf{Hom}}_{\mathrm{ol}}(A\times B, A'\times B')$, where $q_1 \colon A\times B \to A$ and $q_2 \colon A\times B \to B$ are the two projections, and since

$$(u *_0 q_1, v *_0 q_2) *_0 p = (u *_0 q_1 *_0 p, v *_0 q_2 *_0 p) = (u *_0 p_1, v *_0 p_2)\,,$$

the square indeed commutes. □

6. Strict functorialities of the comma construction

The purpose of this section is to study the functorialities of the comma construction when restricted to higher strict transformations.

6.1. — Fix C an ω-category. The inclusion $\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ induces inclusions

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \quad\text{and}\quad \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C\,.$$

In particular, we can restrict the Gray ω-functor

$$-\downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$$

to a Gray ω-functor

$$-\downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

The goal of what follows is to prove that this Gray ω-functor actually lands in $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$.

The strategy is obvious. We gave in 5.2 a formula for this Gray ω-functor and it suffices to check that the formula defines a cell of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$. But this formula involves the oplax transformation γ of the universal 2-square which does not live in $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$!

Nevertheless, we will see that the result is indeed in $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$. To do so, we will introduce an intermediate slice ω-category

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$$

based on the inclusions

$$\Gamma\underline{\mathsf{Hom}}(A,B) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)$$

of 2.20.

6.2. — Let A and B be two ω-categories. As mentioned above, we defined in 2.20 inclusions

$$\Gamma\underline{\mathsf{Hom}}(A,B) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)$$

factorizing the canonical inclusion.

If B is fixed and A varies, we get inclusions

$$\Gamma\underline{\mathsf{Hom}}(-,B) \hookrightarrow \underline{\mathsf{Hom}}(-,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(-,B)$$

of anti Gray ω-functors from $(\omega\text{-}\mathcal{C}at_{\mathrm{cart}})^{\mathrm{t}}$ to $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, and hence by Proposition 1.10, inclusions of right sub-$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$-modules. In particular, this means that if α is a cell of $\underline{\mathsf{Hom}}(A',\Gamma B)$ and u is a cell of $\underline{\mathsf{Hom}}(A,A')$, then $\alpha *_r u$, where $*_r$ denotes the right action of 4.2, is a cell of $\underline{\mathsf{Hom}}(A,\Gamma B)$.

Moreover, for the same reasons as in 4.5, each of the three ω-categories

$$\Gamma\underline{\mathsf{Hom}}(A,B) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)$$

is the object of morphisms of a category internal to $\omega\text{-}\mathcal{C}at$ and, by naturality, the two inclusions are morphisms of internal categories. In particular, if α and β are k-cells of $\underline{\mathsf{Hom}}(A,\Gamma B)$ such that $\mathbb{t}(\beta) = \mathbb{s}(\alpha)$, then $\beta *_c \alpha$, where $*_c$ denotes the internal composition of cylinders of 2.9, is a k-cell of $\underline{\mathsf{Hom}}(A,\Gamma B)$.

6.3. — Fix C an ω-category. We define an ω-category $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ in the following way:

The objects of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ are the same as the ones of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C$ or of $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$, that is, ω-categories A endowed with an ω-functor $f\colon A \to C$.

If $(A, f\colon A \to C)$ and $(A', f'\colon A' \to C)$ are two such objects, we set

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at/_{\Gamma}C}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}(A,A') \times_{\underline{\mathsf{Hom}}(A,C)} \underline{\mathsf{Hom}}(A,\Gamma C) \times_{\underline{\mathsf{Hom}}(A,C)} \{f\}\,.\end{aligned}$$

For the moment, $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ has only been defined as a graph enriched in $\omega\text{-}\mathcal{C}at$.

Recall that

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}(A,A') \times_{\underline{\mathsf{Hom}}(A,C)} \Gamma\underline{\mathsf{Hom}}(A,C) \times_{\underline{\mathsf{Hom}}(A,C)} \{f\}\end{aligned}$$

and

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,A') \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)} \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)} \{f\}\,.\end{aligned}$$

Therefore, the monomorphisms

$$\Gamma\underline{\mathsf{Hom}}(A,C) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma C) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\text{oplax}}(A,C)\,,$$

as they are compatible with the source and target operations of internal categories, induce monomorphisms between these fiber products and hence monomorphisms of graphs enriched in $\omega\text{-}\mathcal{C}at$

$$\omega\text{-}\mathcal{C}at_{\text{cart}}/C \hookrightarrow \omega\text{-}\mathcal{C}at_{\text{cart}}/_{\Gamma}C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C\,.$$

We will consider these monomorphisms as inclusions. Note that $\omega\text{-}\mathcal{C}at_{\text{cart}}/_{\Gamma}C$ have not only the same objects as $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C$ but also the same 1-cells. (But their k-cells differ for $k > 1$.)

Proposition 6.4. — *If C is an ω-category, then $\omega\text{-}\mathcal{C}at_{\text{cart}}/_{\Gamma}C$ is a sub-Gray ω-category of $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C$. It is actually a strict ω-category.*

Proof. — Let $(A, f\colon A \to C)$, $(A', f'\colon A' \to C)$ and $(A'', f''\colon A'' \to C)$ be three objects of $\omega\text{-}\mathcal{C}at/_{\Gamma}C$ and let (u,α) be a cell of $\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at/_{\Gamma}C}((A,f),(A',f'))$ and (u',α') a cell of $\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at/_{\Gamma}C}((A',f'),(A'',f''))$. By definition, their composition in $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C$ is given by

$$(u',\alpha') *_0 (u,\alpha) = \big(u' *_0 u, \alpha *_c (\alpha' *_r u)\big)\,,$$

where $*_r$ denotes the right action of 4.2 and $*_c$ denotes the internal composition on $\Gamma\underline{\mathsf{Hom}}_{\text{oplax}}(A,C)$ of 2.9. Since $\omega\text{-}\mathcal{C}at_{\text{cart}}$ is a sub-Gray ω-category of $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$, the cell $u' *_0 u$ lives in $\omega\text{-}\mathcal{C}at_{\text{cart}}$. Moreover, by 6.2, the cell $\alpha' *_r u$ is in $\underline{\mathsf{Hom}}(A,\Gamma C)$ and hence so is $\alpha *_c (\alpha' *_r u)$, thereby proving the stability under composition of $\omega\text{-}\mathcal{C}at_{\text{cart}}/_{\Gamma}C$. The compatibility with units is obvious.

The fact that $\omega\text{-}\mathcal{C}at_{\text{cart}}/_{\Gamma}C$ is a strict ω-category follows from the formula giving the composition and the fact that $\omega\text{-}\mathcal{C}at_{\text{cart}}$ is a strict ω-category. □

6.5. — Let C be an ω-category. One defines similarly an ω-category $\omega\text{-}\mathcal{C}at_{\text{cart}}\overset{\text{to}}{/_{\Gamma}}C$ with Gray inclusions

$$\omega\text{-}\mathcal{C}at_{\text{cart}}\overset{\text{to}}{/}C \hookrightarrow \omega\text{-}\mathcal{C}at_{\text{cart}}\overset{\text{to}}{/_{\Gamma}}C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\overset{\text{to}}{/}C\,.$$

The objects of $\omega\text{-}\mathcal{C}at_{\text{cart}}\overset{\text{to}}{/_{\Gamma}}C$ are the ω-categories B endowed with an ω-functor $g\colon B \to C$, and if $(B, g\colon B \to C)$ and $(B', g'\colon B' \to C)$ are two such objects, we have

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at\overset{\text{to}}{/_{\Gamma}}C}((B,g),(B',g'))\\ &\qquad = \{g\} \times_{\underline{\mathsf{Hom}}(B,C)} \underline{\mathsf{Hom}}(B,\Gamma C) \times_{\underline{\mathsf{Hom}}(B,C)} \underline{\mathsf{Hom}}(B,B')\,.\end{aligned}$$

Proposition 6.6. — *Let C be an ω-category. The Gray ω-functor*

$$-\downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\overset{\text{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$$

induces an ω-functor

$$-\downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\text{cart}}/_{\Gamma}C \times \omega\text{-}\mathcal{C}at_{\text{cart}}\overset{\text{to}}{/_{\Gamma}}C \to \omega\text{-}\mathcal{C}at_{\text{cart}}$$

and in particular an ω*-functor*

$$- \downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Proof. — Consider A, A', B and B' four ω-categories over C. We have to show that the composite ω-functor

$$\begin{array}{c}
\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at_{\mathrm{cart}/_{\Gamma}} C}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/_{\Gamma}} C}((B,g),(B',g')) \\
\big\downarrow \\
\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C}((B,g),(B',g')) \\
\big\downarrow{\scriptstyle - \downarrow_C -} \\
\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, B') \\
\simeq \\
\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A' \downarrow_C B')
\end{array}$$

factors through

$$\underline{\mathsf{Hom}}(A \downarrow_C B, A' \downarrow_C B')\,.$$

Using 5.2, and with its notation, this ω-functor is given on k-cells by

$$(u, \alpha, \beta, v) \mapsto (u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)\,,$$

where

$$(u, \alpha, \beta, v) \ \text{ is in } \ \underline{\mathsf{Hom}}(A, A')_k \times \underline{\mathsf{Hom}}(A, \Gamma C)_k \times \underline{\mathsf{Hom}}(B, \Gamma C)_k \times \underline{\mathsf{Hom}}(B, B')_k\,.$$

Note that γ can be seen as a 0-cell of $\underline{\mathsf{Hom}}(A \downarrow_C B, \Gamma C)$. Using 6.2, we get that

$$(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)$$

is actually a k-cell in $\underline{\mathsf{Hom}}(A \downarrow_C B, \Gamma C)$, and in particular,

$$(u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)$$

is a k-cell of

$$\underline{\mathsf{Hom}}(A \downarrow_C B, A') \underset{\underline{\mathsf{Hom}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}(A \downarrow_C B, \Gamma C) \underset{\underline{\mathsf{Hom}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}(A \downarrow_C B, B')\,,$$

which is canonically isomorphic to

$$\underline{\mathsf{Hom}}(A \downarrow_C B, A' \times_C \Gamma C \times_C B') \simeq \underline{\mathsf{Hom}}(A \downarrow_C B, A' \downarrow_C B')\,.$$

As this canonical isomorphism is compatible with the canonical isomorphism between

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, \Gamma C) \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, B')$$

and

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A' \times_C \Gamma C \times_C B') \simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A' \downarrow_C B')\,,$$

this proves the result. □

Remark 6.7. — As in Remark 5.5, if A and B are two fixed ω-categories, there is a canonical embedding

$$\underline{\mathsf{Hom}}(A,C)^{\mathrm{o}} \times \underline{\mathsf{Hom}}(B,C)^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}/}C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/}C$$

and the comma construction thus restricts to an ω-functor

$$-\downarrow_C -\colon\ \underline{\mathsf{Hom}}(A,C)^{\mathrm{o}} \times \underline{\mathsf{Hom}}(B,C)^{\mathrm{to}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Similarly, if A is an ω-category, let us denote by $\underline{\mathsf{Hom}}_{\Gamma}(A,B)$ the total dual of the fiber at A of the forgetful ω-functor

$$U\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}/_{\Gamma}}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

so that we have inclusions

$$\underline{\mathsf{Hom}}(A,C) \hookrightarrow \underline{\mathsf{Hom}}_{\Gamma}(A,C) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)\,.$$

By definition (and duality), we have a canonical embedding

$$\underline{\mathsf{Hom}}_{\Gamma}(A,C)^{\mathrm{o}} \times \underline{\mathsf{Hom}}_{\Gamma}(B,C)^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}/}C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/}C$$

and the comma construction also restricts to an ω-functor

$$-\downarrow_C -\colon\ \underline{\mathsf{Hom}}_{\Gamma}(A,C)^{\mathrm{o}} \times \underline{\mathsf{Hom}}_{\Gamma}(B,C)^{\mathrm{to}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Corollary 6.8. — *If $B \to C$ is an ω-functor, then we have an ω-functor*

$$-\downarrow_C B\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}/_{\Gamma}}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$$

and in particular an ω-functor

$$-\downarrow_C B\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}/}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

and if $A \to C$ is an ω-functor, then we have an ω-functor

$$A\downarrow_C -\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/_{\Gamma}}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$$

and in particular an ω-functor

$$A\downarrow_C -\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Remark 6.9. — By duality (see Remark 5.6), Proposition 6.6 implies that the lax comma construction induces an ω-functor

$$-\downarrow'_C -\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{top}}{/}C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{co}}{/}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

and, more generally, an ω-functor

$$-\downarrow'_C -\colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{top}}{/_{\Gamma'}}C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{co}}{/_{\Gamma'}}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

where the ω-categories

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{top}}{/_{\Gamma'}}C \quad\text{and}\quad \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{co}}{/_{\Gamma'}}C$$

are defined from their undecorated-by-Γ' analogue by replacing, in the definition of the ω-category of morphisms, $\Gamma'\,\underline{\mathsf{Hom}}(-,C)$ by $\underline{\mathsf{Hom}}(-,\Gamma'C)$. (Recall that we set $\Gamma'X = \underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1,X)$.)

7. Application: Grothendieck construction for ω-categories

Our main motivation for studying the functorialities of the comma construction was the Grothendieck construction for ω-categories, to which we will devote a separate paper [**2**]. In this short final section, we define the Grothendieck construction for ω-categories in terms of comma ω-categories and we deduce functoriality results for the Grothendieck construction.

7.1. — Let I be an ω-category and let $F\colon I^o \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ be an ω-functor. We define the *(contravariant) Grothendieck construction* $\int_I F$ of F to be the total dual of the comma construction of the diagram

$$\mathrm{D}_0 \xrightarrow{\mathrm{D}_0} \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \xleftarrow{F} I^o \quad ,$$

where the left arrow corresponds to the object D_0 of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$. In other words, we have

$$\textstyle\int_I F = \big(\mathrm{D}_0 \downarrow_{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}} F\big)^o .$$

The second projection of the comma construction induces an ω-functor $p\colon \int_I F \to I$.

Remark 7.2. — Although the ω-category $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is not small, the comma construction $\mathrm{D}_0 \downarrow F$ makes sense and is a small ω-category.

Remark 7.3. — By Example 3.2, the Grothendieck construction of an ω-functor

$$F\colon I^o \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$$

is the total dual of a relative slice:

$$\textstyle\int_I F = \big(\mathrm{D}_0 \downarrow F\big)^o = \big(\mathrm{D}_0\backslash I^o\big)^o .$$

In other words, we have a pullback square

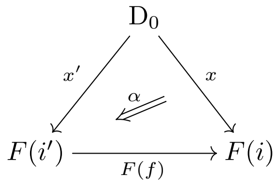
$$\begin{array}{ccc}
(\int_I F)^o & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \\
\big\downarrow & \lrcorner & \big\downarrow U \\
I^o & \xrightarrow[F]{} & \omega\text{-}\mathcal{C}at_{\mathrm{cart}}
\end{array} \quad ,$$

where U denotes the forgetful ω-functor.

7.4. — Let $F\colon I^o \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ be an ω-functor. To convince ourselves that our definition of the Grothendieck construction is reasonable, let us concretely describe the cells of $\int_I F$ in low dimensions.

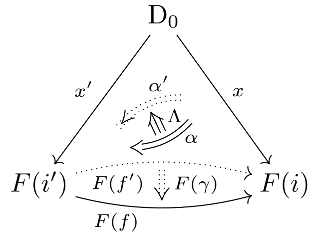
By definition, an object of $\int_I F$ corresponds to an object i of I and an ω-functor $x\colon \mathrm{D}_0 \to F(i)$. These objects can thus be identified with pairs (i, x), where i is an object of I and x an object of $F(i)$.

A 1-cell of $\int_I F$ corresponds to a 1-cell $f\colon i \to i'$ of I and a 2-triangle



in $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$. But the data of such an oplax transformation α is equivalent to the data of a 1-cell $\alpha\colon x \to F(f)(x')$ of $F(i)$. The 1-cells from (i, x) to (i', x') can thus be identified with pairs $(f\colon i \to i', \alpha\colon x \to F(f)(x'))$.

A 2-cell of $\int_I F$ corresponds to a 2-cell $\gamma\colon f \Rightarrow f'\colon i \to i'$ of I and a cone



in $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$. But the data of such an oplax 2-transformation Λ is equivalent to the data of a 2-cell $\Lambda\colon \alpha \Rrightarrow F(\gamma)_{x'} *_0 \alpha'$ of $F(i)$. The 2-cells from (f, α) to (f', α') can thus be identified with these pairs (γ, Λ).

Remark 7.5. — The Grothendieck construction for ω-categories was first defined by Warren [**10**] using explicit formulas. In [**2**], we will show that our definition is equivalent to Warren's one (up to some duality, as Warren defines the *covariant* Grothendieck construction).

7.6. — We will denote by $\omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}}$ the (very large) Gray ω-category of possibly large ω-categories, ω-functors, oplax transformations and higher oplax transformations between them. We have a fully faithful inclusion $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}} \hookrightarrow \omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}}$. The ω-category $\omega\text{-}\mathcal{C}at_{\text{cart}}$ is an object of $\omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}}$. When we consider $\omega\text{-}\mathcal{C}at_{\text{cart}}$ as an object of $\omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}}$, we will denote it by $\{\omega\text{-}\mathcal{C}at_{\text{cart}}\}$. In particular, we will write

$$\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}} \overset{\text{to}}{/} \{\omega\text{-}\mathcal{C}at_{\text{cart}}\}$$

for the Gray ω-category defined by the pullback

$$\begin{array}{ccc} \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}} \overset{\text{to}}{/} \{\omega\text{-}\mathcal{C}at_{\text{cart}}\} & \longrightarrow & \omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}} \overset{\text{to}}{/} \{\omega\text{-}\mathcal{C}at_{\text{cart}}\} \\ \downarrow & \lrcorner & \downarrow U \\ \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}} & \hookrightarrow & \omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}} \end{array},$$

where $\omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}} \overset{\text{to}}{/} \{\omega\text{-}\mathcal{C}at_{\text{cart}}\}$ is one of the (very large) slice Gray ω-categories of 4.18 and U is the forgetful Gray ω-functor.

Theorem 7.7. — *The Grothendieck construction defines a Gray ω-functor*

$$\begin{array}{rcl} \textstyle\int \colon \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} & \to & \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \\ F\colon I^{\mathrm{o}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}} & \mapsto & \int_I F \end{array} .$$

Proof. — The Grothendieck construction factors as

$$\big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \xrightarrow{\mathrm{D}_0 \downarrow -} \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\big)^{\mathrm{to}} \xrightarrow{D_{\mathrm{o}}} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} ,$$

where D_{o} is the total dual. But the left arrow is a Gray ω-functor by Theorem 5.3 (and more precisely Corollary 5.4) and the right arrow is a Gray ω-functor by 2.27. ☐

Proposition 7.8. — *The ω-functor $p\colon \int_I F \to I$ is natural in $F\colon I^{\mathrm{o}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ in the sense that it defines a strict transformation from the Gray ω-functor*

$$\textstyle\int \colon \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$$

to the Gray ω-functor obtained by composing

$$\big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \xrightarrow{U} \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\big)^{\mathrm{to}} \xrightarrow{D_{\mathrm{o}}} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} ,$$

where D_o is the total dual.

Proof. — This is a particular case of the analogous result for the comma construction, that is, Proposition 5.8. ☐

Proposition 7.9. — *If I is a fixed ω-category, the Grothendieck construction restricts to a Gray ω-functor*

$$\textstyle\int_I \colon \underline{\mathsf{Hom}}_{\mathrm{oplax}}(I^{\mathrm{o}}, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} .$$

Proof. — By Remark 5.5, there is a canonical embedding

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(I^{\mathrm{o}}, \omega\text{-}\mathcal{C}at_{\mathrm{cart}})^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\} ,$$

hence the result by the previous proposition. ☐

7.10. — We are now going to state the strict functorialities of the Grothendieck construction. We can define, as in 7.6, a large strict ω-category

$$\begin{array}{ccc} \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\} & \longrightarrow & \omega\text{-}\mathcal{C}\mathrm{AT}_{\mathrm{cart}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\} \\ \big\downarrow & \lrcorner & \big\downarrow U \\ \omega\text{-}\mathcal{C}at_{\mathrm{cart}} & \hookrightarrow & \omega\text{-}\mathcal{C}\mathrm{AT}_{\mathrm{cart}} \end{array} ,$$

where $\omega\text{-}\mathcal{C}\mathrm{AT}_{\mathrm{cart}}$ is the (very large) strict ω-category of possibly large ω-categories, ω-functors, strict transformations and higher strict transformations between them.

Proposition 7.11. — *The Grothendieck construction restricts to a strict ω-functor*

$$\textstyle\int\colon \big(\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

and, if I is a fixed ω-category, to a strict ω-functor

$$\textstyle\int_I\colon\ \underline{\mathsf{Hom}}(I^{\mathrm{o}}, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Proof. — This follows from the analogous result for the comma construction, that is, Proposition 6.6. ☐

Remark 7.12. — By Remark 6.7 and using the same notation, the second ω-functor of the above proposition actually extends to an ω-functor

$$\textstyle\int_I\colon\ \underline{\mathsf{Hom}}_{\Gamma}(I^{\mathrm{o}}, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

We end the paper by an opening on a definition of the Grothendieck construction for (anti) Gray ω-functors, based on Remark 7.3.

7.13. — Let $\mathbb{I}$ be a Gray ω-category, so that $\mathbb{I}^{\mathrm{o}}$ is an anti Gray ω-category, and fix $F\colon \mathbb{I}^{\mathrm{o}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ an anti Gray ω-functor. We define the *Grothendieck construction* $\int_{\mathbb{I}} F$ of F as the Gray ω-category obtained by taking the total dual of the pullback of anti Gray ω-categories

$$\begin{array}{ccc}
(\int_{\mathbb{I}} F)^{\mathrm{o}} & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \\
\big\downarrow & \lrcorner & \big\downarrow U \\
\mathbb{I}^{\mathrm{o}} & \xrightarrow[F]{} & \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}
\end{array}\quad,$$

where $\mathrm{D}_0\backslash\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ is one of the slice anti Gray ω-categories defined in 4.18 and U is the forgetful anti Gray ω-functor.

In the case where $\mathbb{I}$ comes from a strict ω-category I and the ω-functor F factors through the inclusion $\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, we recover the Grothendieck construction defined in 7.1. Indeed, consider the commutative diagram

$$\begin{array}{ccccc}
(\int_I F)^{\mathrm{o}} & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathcal{C}at_{\mathrm{cart}} & \hookrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \\
\big\downarrow & & \big\downarrow U & & \big\downarrow U \\
I^{\mathrm{o}} & \xrightarrow[F]{} & \omega\text{-}\mathcal{C}at_{\mathrm{cart}} & \hookrightarrow & \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}
\end{array}\quad.$$

The left square is cartesian by Remark 7.3. As for the right square, it is easily seen to be cartesian as well using the canonical isomorphism $\underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, C) \simeq \underline{\mathsf{Hom}}(\mathrm{D}_0, C)$, for any ω-category C. It follows that the composite of these two squares is cartesian, showing the compatibility of the two Grothendieck constructions.

In future work, we plan to investigate this Grothendieck construction for (anti) Gray ω-functors.

Appendix A. Non-existence of the Gray ω-category of cylinders

The purpose of this appendix is to explain why, for $\mathbb{C}$ a Gray ω-category, there is no natural Gray ω-category of cylinders $\Gamma\mathbb{C}$.

A.1. — By definition, a k-cylinder in an ω-category C is an ω-functor $\mathrm{D}_1 \otimes \mathrm{D}_k \to C$. This definition extends immediately to Gray ω-categories: if $\mathbb{C}$ is a Gray ω-category, a k-cylinder in $\mathbb{C}$ is a Gray ω-functor $\mathrm{D}_1 \otimes \mathrm{D}_k \to \mathbb{C}$, where $\mathrm{D}_1 \otimes \mathrm{D}_k$ is considered as a Gray ω-category. Moreover, the operations source, target and units of cylinders are still defined and if $\mathbb{C}$ is a Gray ω-category we get a structure of reflexive ω-graph on cylinders in $\mathbb{C}$, extending the one of ΓC when $\mathbb{C}$ comes from an ω-category C.

A.2. — The combinatorial description of k-cylinders (see for instance [**4**, Proposition B.1.6]) still applies in the Gray world. In other words, giving a k-cylinder in a Gray ω-category $\mathbb{C}$ amounts to the following data:

- two k-cells x and y of $\mathbb{C}$,
- for every $0 \leqslant l < k$, two $(k+1)$-cells

$$\gamma_l^- : \gamma_{l-1}^+ *_{l-1} \dots *_1 \gamma_0^+ *_0 s_l(x) \to s_l(y) *_0 \gamma_0^- *_1 \cdots *_{l-1} \gamma_{l-1}^-$$
$$\gamma_l^+ : \gamma_{l-1}^+ *_{l-1} \dots *_1 \gamma_0^+ *_0 t_l(x) \to t_l(y) *_0 \gamma_0^- *_1 \cdots *_{l-1} \gamma_{l-1}^-$$

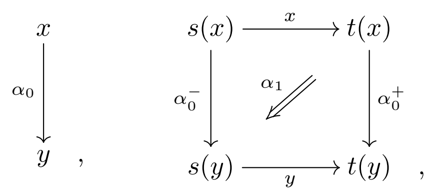
  of $\mathbb{C}$,
- a $(k+1)$-cell

$$\gamma_k \colon \gamma_{k-1}^+ *_{k-1} \dots *_1 \gamma_0^+ *_0 x \to y *_0 \gamma_0^- *_1 \cdots *_{k-1} \gamma_{k-1}^-$$

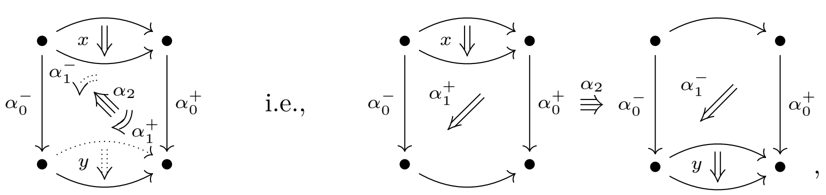
  of $\mathbb{C}$.

In the above formulas we use the convention that $*_i$ has priority over $*_j$ whenever $i < j$.

In particular, 0-cylinders are 1-cells of $\mathbb{C}$, 1-cylinders are 2-squares in $\mathbb{C}$,



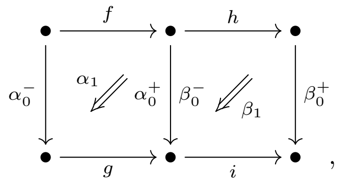
and 2-cylinders are diagrams in $\mathbb{C}$ of the shape



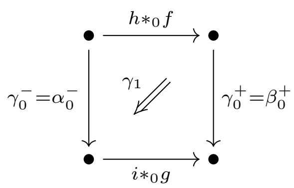
with

$$\alpha_2 \colon \alpha_1^+ *_1 \alpha_0^+ *_0 x \to y *_0 \alpha_0^- *_1 \alpha_1^- \,.$$

A.3. — If $\mathbb{C}$ is a Gray ω-category (or more generally a sesquicategory), 0-cylinders and 1-cylinders in $\mathbb{C}$ organize themselves as a 1-category. Indeed, if α and β are two 1-cylinders in $\mathbb{C}$ that are composable,



so that $\alpha_0^+ = \beta_0^-$, then their composite $\gamma = \beta *_0 \alpha$ is defined as in the strict case, that is, by



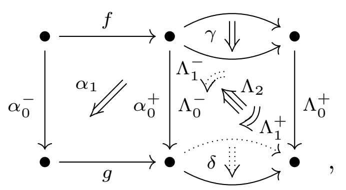
with

$$\gamma_1 = (i *_0 \alpha_1) *_1 (\beta_1 *_0 f)\,.$$

It is immediate that this indeed defines a 1-cylinder with the appropriate source and target, and that 0-cylinders and 1-cylinders in $\mathbb{C}$, together with this composition, form a 1-category.

We will now explain why this 1-category does not extend to higher dimensions.

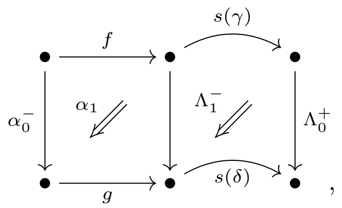
A.4. — Let $\mathbb{C}$ be a Gray ω-category and consider a 1-cylinder α and a 2-cylinder Λ in $\mathbb{C}$ such that $t_0(\alpha) = s_0(\Lambda)$, that is, a diagram



with $\alpha_0^+ = \Lambda_0^-$. Let us try to define the composite $\Gamma = \Lambda *_0 \alpha$. This 2-cylinder Γ must satisfy

$$s(\Gamma) = s(\Lambda *_0 \alpha) = s(\Lambda) *_0 \alpha \quad \text{and} \quad t(\Gamma) = t(\Lambda *_0 \alpha) = t(\Lambda) *_0 \alpha\,,$$

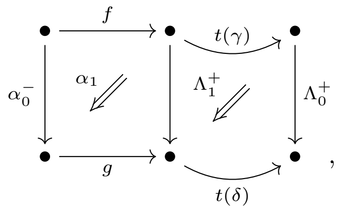
as in any sesquicategory. In other words, its "back face" has to be the composite of



and so

$$\Gamma_1^- = (s(\delta) *_0 \alpha_1) *_1 (\Lambda_1^- *_0 f)\,,$$

and its "front face" the composite of



and so

$$\Gamma_1^+ = (t(\delta) *_0 \alpha_1) *_1 (\Lambda_1^+ *_0 f)\,.$$

In particular,

$$\Gamma_0^- = \alpha_0^- \quad \text{and} \quad \Gamma_0^+ = \Lambda_0^+\,.$$

Moreover, to be compatible with the strict case, the "top face" of Γ has to be $\gamma *_0 f$ and its "bottom face" $\delta *_0 g$.

Therefore, all the defining data of Γ are already specified, except Γ_2. Now, by definition of a 2-cylinder, we have

$$\begin{aligned} s(\Gamma_2) &= \Gamma_1^+ *_1 \Gamma_0^+ *_0 (\gamma *_0 f) \\ &= \big[(t(\delta) *_0 \alpha_1) *_1 (\Lambda_1^+ *_0 f)\big] *_1 \Lambda_0^+ *_0 (\gamma *_0 f) \\ &= (t(\delta) *_0 \alpha_1) *_1 (\Lambda_1^+ *_0 f) *_1 (\Lambda_0^+ *_0 \gamma *_0 f) \\ &= (t(\delta) *_0 \alpha_1) *_1 \big[(\Lambda_1^+ *_1 (\Lambda_0^+ *_0 \gamma)) *_0 f\big] \\ &= (t(\delta) *_0 \alpha_1) *_1 (s(\Lambda_2) *_0 f) \end{aligned}$$

and

$$\begin{aligned} t(\Gamma_2) &= (\delta *_0 g) *_0 \Gamma_0^- *_1 \Gamma_1^- \\ &= (\delta *_0 g) *_0 \alpha_0^- *_1 \big[(s(\delta) *_0 \alpha_1) *_1 (\Lambda_1^- *_0 f)\big] \\ &= \big[(\delta *_0 (g *_0 \alpha_0^-) *_1 (s(\delta) *_0 \alpha_1)\big] *_1 (\Lambda_1^- *_0 f) \\ &= \big[(\delta *_0 t(\alpha_1)) *_1 (s(\delta) *_0 \alpha_1)\big] *_1 (\Lambda_1^- *_0 f)\,. \end{aligned}$$

In a strict ω-category, we could apply the exchange rule to carry on this computation but in a Gray ω-category all we have is a 3-cell

$$\delta *_0 \alpha_1 \colon (\delta *_0 t(\alpha_1)) *_1 (s(\delta) *_0 \alpha_1) \Rrightarrow (t(\delta) *_0 \alpha_1) *_1 (\delta *_0 s(\alpha_1))\,.$$

We thus get
$$t(\Gamma_2) = s(\delta *_0 \alpha_1) *_1 (\Lambda_1^- *_0 f)$$
and
$$\begin{aligned} t(\delta *_0 \alpha_1) *_1 (\Lambda_1^- *_0 f) &= \big[(t(\delta) *_0 \alpha_1) *_1 (\delta *_0 s(\alpha_1))\big] *_1 (\Lambda_1^- *_0 f) \\ &= (t(\delta) *_0 \alpha_1) *_1 (\delta *_0 \alpha_0^+ *_0 f) *_1 (\Lambda_1^- *_0 f) \\ &= (t(\delta) *_0 \alpha_1) *_1 \big[((\delta *_0 \alpha_0^+) *_1 \Lambda_1^-) *_0 f\big] \\ &= (t(\delta) *_0 \alpha_1) *_1 \big[((\delta *_0 \Lambda_0^-) *_1 \Lambda_1^-) *_0 f\big] \\ &= (t(\delta) *_0 \alpha_1) *_1 (t(\Lambda_2) *_0 f)\,. \end{aligned}$$
Therefore we have a zigzag of 3-cells
$$s(\Gamma_2) \Rrightarrow t(\delta *_0 \alpha_1) *_1 (\Lambda_1^- *_0 f) \Lleftarrow t(\Gamma_2)\,,$$
the left and right 3-cells being
$$(t(\delta) *_0 \alpha_1) *_1 (\Lambda_2 *_0 f) \quad \text{and} \quad (\delta *_0 \alpha_1) *_1 (\Lambda_1^- *_0 f)\,,$$
respectively. In general, we cannot compose this zigzag and there is no 3-cell Γ_2 with the correct source and target.

Remark A.5. — In an anti Gray ω-category, the orientation of the 3-cell $\delta *_0 \alpha_1$ would be reverted so that we get two composable 3-cells
$$s(\Gamma_2) \Rrightarrow s(\delta *_0 \alpha_1) *_1 (\Lambda_1^- *_0 f) \Rrightarrow t(\Gamma_2)\,,$$
allowing to define Γ_2. This means that we can make sense of the 2-cylinder $\Lambda *_0 \alpha$ in an anti Gray ω-category.

Dually, one can make sense of the composite $\beta *_0 \Lambda$, when β is a 1-cylinder, Λ a 2-cylinder and $t_0(\Lambda) = s_0(\beta)$, in a Gray ω-category but not in an anti Gray ω-category.

Overall, one cannot make sense of the composite $\beta *_0 \Lambda *_0 \alpha$ in a Gray ω-category or in an anti Gray ω-category.

References

[1] D. Ara – "A Quillen Theorem B for strict ∞-categories", *J. Lond. Math. Soc. (2)* **100** (2019), no. 2, p. 470–497.

[2] D. Ara, A. Gagna & L. Guetta – "Homotopical properties of the Grothendieck construction for strict ω-categories", work in progress.

[3] D. Ara & G. Maltsiniotis – "Un théorème A de Quillen pour les ∞-catégories strictes II : la preuve ∞-catégorique", *High. Struct.* **4** (2020), no. 1, p. 284–388.

[4] ———, "Joint et tranches pour les ∞-catégories strictes", *Mém. Soc. Math. Fr. (N.S.)* **165** (2020).

[5] D. Gepner & H. Heine – "Oriented category theory", Preprint, arXiv:2510.10504, 2025.

[6] Y. Lafont & F. Métayer – "Polygraphic resolutions and homology of monoids", *J. Pure Appl. Algebra* **213** (2009), no. 6, p. 947–968.

[7] Y. Lafont, F. Métayer & K. Worytkiewicz – "A folk model structure on omega-cat", *Adv. Math.* **224** (2010), no. 3, p. 1183–1231.

[8] F. Loubaton – "Categorical theory of (∞,ω)-categories", Preprint, arXiv:2406.05425, 2024.

[9] R. W. Thomason – "Homotopy colimits in the category of small categories", *Math. Proc. Cambridge Philos. Soc.* **85** (1979), no. 1, p. 91–109.

[10] M. A. Warren – "The strict ω-groupoid interpretation of type theory", in *Models, logics, and higher-dimensional categories: A tribute to the work of Mihály Makkai*, American Mathematical Society, 2011, p. 291–340.

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