ON AUTOEQUIVALENCES OF THE $(\infty, 1)$-CATEGORY OF ∞-OPERADS

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Abstract. We study the $(\infty, 1)$-category of autoequivalences of ∞-operads. Using techniques introduced by Toën, Lurie, and Barwick and Schommer-Pries, we prove that this $(\infty, 1)$-category is a contractible ∞-groupoid. Our calculation is based on the model of complete dendroidal Segal spaces introduced by Cisinski and Moerdijk. Similarly, we prove that the $(\infty, 1)$-category of autoequivalences of non-symmetric ∞-operads is the discrete monoidal category associated to $\mathbf{Z}/2\mathbf{Z}$. We also include a computation of the $(\infty, 1)$-category of autoequivalences of (∞, n)-categories based on Rezk's Θ_n-spaces.

Contents

1. Introduction

Higher category theory and higher operad theory can be formalized by means of a plethora of different explicit approaches, all of them having their merits and drawbacks. These theories have applications in fields as diverse as algebraic topology, (derived) algebraic geometry, representation theory and homological algebra; see for instance the foundational work of Toën–Vezzosi [TV04, TV05, TV08] and Lurie [Lur09a, Lur12, Lur13]. Having a specific problem at hand, we are hence able to choose an approach accordingly, and it is thus important for practical and theoretical purposes to know how to compare these different formulations.

In the case of $(\infty, 1)$-categories, the state of the art is very satisfactory. By now, there are many different approaches to the theory of $(\infty, 1)$-categories, including quasi-categories [Joy02], simplicial categories [Ber07], Segal categories [HS01] and complete Segal spaces [Rez01]. Each of these theories is organized in a Quillen model category and these are related by a web of Quillen equivalences; see [Ber10] for a survey on these Quillen equivalences.

In [Toë05], Toën took this one step further and, based on earlier work of Simpson [Sim01], offered an axiomatization of the theory of $(\infty, 1)$-categories. Moreover, he showed that the $(\infty, 1)$-category of autoequivalences of one such theory is the discrete category on the cyclic group $\mathbf{Z}/2\mathbf{Z}$ of order two, the non-trivial element being the passage to the opposite $(\infty,1)$-category. More precisely, he computed the (derived) autoequivalences of the simplicial category obtained as the Dwyer–Kan localization of complete Segal spaces. A similar calculation was given by Lurie in [Lur09b, Section 4.4] using the language of quasi-categories. These computations imply that any two possibly different ways of comparing two models for $(\infty,1)$-categories differ at most by the passage to opposites.

In this paper, we study the $(\infty,1)$-category of autoequivalences of ∞-operads. As in the case of $(\infty,1)$-categories, there are many different approaches to ∞-operads, including simplicial operads [CM13b], ∞-operads in the sense of Lurie [Lur13, Chapter 2], dendroidal sets [MW07, CM11] and complete dendroidal Segal spaces [CM13a] (which will be called Ω-spaces in this paper). Again, there are Quillen model categories in the background, and thanks to recent work of Cisinski–Moerdijk [CM11, CM13a, CM13b] and Heuts–Hinich–Moerdijk [HHM15], it is known that all these model structures are connected by Quillen equivalences.

We show that the $(\infty,1)$-category of autoequivalences of the $(\infty,1)$-category of ∞-operads is a contractible ∞-groupoid. More precisely, we prove that the quasi-category of autoequivalences of Ω-spaces is a contractible Kan complex. This implies that if there is a way to compare two models for ∞-operads, then this can be done in an essentially unique way. Similarly, we show that the $(\infty,1)$-category of autoequivalences of the $(\infty,1)$-category of non-symmetric ∞-operads is the discrete category on the cyclic group $\mathbf{Z}/2\mathbf{Z}$ of order two, the non-trivial element being the "mirror autoequivalence".

One general strategy to compute the autoequivalences of an $(\infty,1)$-category $\mathcal{C}$, following Toën [Toë05], Lurie [Lur09b, Section 4.4] and Barwick–Schommer-Pries [BSP13], is the following. One first identifies a small category A inside $\mathcal{C}$ such that

(i) the inclusion functor $A \hookrightarrow \mathcal{C}$ is dense;
(ii) the autoequivalences of $\mathcal{C}$ restrict to autoequivalences of A.

It then follows formally that the autoequivalences of $\mathcal{C}$ sit fully faithfully in the autoequivalences of A. The problem is thus reduced to computing the autoequivalences of A (up to the question of essential surjectivity, which is easy in our cases).

If the $(\infty,1)$-category $\mathcal{C}$ is a localization of an $(\infty,1)$-category $\mathcal{P}(A)$ of simplicial presheaves on a small category A, then one might hope that, in good cases, A would satisfy the two conditions above. However, in practice it is hard to show the second point directly. For this purpose, Toën introduced the idea of using an intermediate (large) category: the so-called 0-truncated objects of $\mathcal{C}$. This category is trivially stable under autoequivalences of $\mathcal{C}$. Thus, if we assume that the objects of A are 0-truncated, the verification of the second point is reduced to showing that autoequivalences of 0-truncated objects of $\mathcal{C}$ fix the small category A. In our cases this turns out to be a much simpler problem. This strategy is formalized by our Proposition 3.8. (For the case where $\mathcal{C}$ is the $(\infty,1)$-category of $(\infty,1)$-categories, Lurie [Lur09b, Section 4.4] uses a similar strategy, considering the category of posets as an intermediate (large) category.)

In order to apply this proposition to compute the autoequivalences of ∞-operads, we need a model defined as a localization of simplicial presheaves. The only model of this kind for ∞-operads proposed so far are the Ω-spaces of Cisinski–Moerdijk [CM13a]. These are defined as a localization of simplicial presheaves on the small category Ω of trees introduced by Moerdijk and Weiss [MW07]. It is not hard to identify the 0-truncated Ω-spaces: they are the so-called rigid (strict) operads, i.e., the operads whose underlying category contains no non-trivial isomorphisms. It thus suffices to show that autoequivalences of the category of rigid operads restrict to the small category Ω and to compute the autoequivalences of Ω. Most of our section on ∞-operads is dedicated to the proofs of these two statements. Although this is not formally needed, we also include a similar computation for the autoequivalences of (strict) operads.

We also compute the autoequivalences of non-symmetric ∞-operads using the obvious planar variant of complete dendroidal Segal spaces. The proofs are quite similar to the symmetric case although the combinatorics differs at some points. The difference is mainly due to the fact that objects of the planar version of Ω have no non-trivial automorphisms.

Finally, we include a calculation of the $(\infty,1)$-category of autoequivalences of (∞,n)-categories. This problem has already been solved by Barwick and Schommer-Pries in [BSP13] using a new model for (∞,n)-categories called Υ_n-spaces. Here we provide an alternative calculation using instead the model of Θ_n-spaces introduced by Rezk in [Rez10a, Rez10b], based on the category Θ_n introduced by Joyal in [Joy97]. Our choice of Θ_n-spaces rather than Υ_n-spaces is dictated by the simpler combinatorics of the category Θ_n.

Organization of the paper. In Section 2, we recall some facts about quasi-categories. In Section 3, we study restriction functors induced by dense functors and we formalize the general strategy for calculating autoequivalences of certain quasi-categories. In Section 4, we prove that the quasi-category of autoequivalences of Θ_n-spaces is the discrete category $(\mathbf{Z}/2\mathbf{Z})^n$. Along the way, we compute the autoequivalences of the categories of strict n-categories, rigid strict n-categories and of the category Θ_n. In Section 5, we prove that the quasi-category of autoequivalences of Ω-spaces is a contractible Kan complex. We also calculate the autoequivalences of the categories of operads, rigid operads and of the category Ω. Finally, in Section 6, we turn to the quasi-category of planar Ω-spaces and show that its quasi-category of autoequivalences is the discrete category $\mathbf{Z}/2\mathbf{Z}$. We also describe the autoequivalences of non-symmetric operads, non-symmetric rigid operads and planar trees.

Notation and terminology. If A is a small category, we will denote by $\mathrm{Pr}(A)$ the category of (set-valued) presheaves on A and by $\mathrm{sPr}(A)$ the category of simplicial presheaves on A. If $\mathcal{C}$ is a category, we will denote by $\mathrm{aut}(\mathcal{C})$ the *set* of autoequivalences of $\mathcal{C}$ and by $\mathrm{Aut}(\mathcal{C})$ the *category* of autoequivalences of $\mathcal{C}$. The set $\mathrm{aut}(\mathcal{C})$ and the category $\mathrm{Aut}(\mathcal{C})$ will sometimes be considered as a monoid and a strict monoidal category, respectively, the additional structure being given by composition. We will say that a morphism of a category (or more generally of a strict n-category) is non-trivial if it is not an identity.

If $\mathcal{C}$ and $\mathcal{D}$ are quasi-categories, we will denote by $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ the quasi-category of functors from $\mathcal{C}$ to $\mathcal{D}$. The full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ spanned by the equivalences will be denoted by $\mathrm{Aut}(\mathcal{C})$.

We will assume for simplicity (contrary to the strong opinions of the first two authors) that our model categories have functorial factorizations. If $\mathcal{M}$ is a model category and S is a class of morphisms of $\mathcal{M}$, we will denote by $S^{-1}\mathcal{M}$ the left Bousfield localization of $\mathcal{M}$ with respect to S (if it exists).

We will neglect the usual set theoretic issues related to category theory. In particular, we will shamelessly apply the nerve functor to non-small categories.

2. Review of quasi-categories

In this section, we recall some facts about quasi-categories, mostly about the relation between model categories and quasi-categories, and about localizations of locally presentable quasi-categories. We assume that the reader is familiar with the basics of the theory of quasi-categories as developed in the foundational work of Joyal [Joy02, Joy08a, Joy08b] and Lurie [Lur09a, Lur13]. For an introduction to this theory emphasizing the philosophy, see [Gro15].

2.1. We will denote by $N\colon \mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{Set}$ the nerve functor from (small) categories to simplicial sets. Since the nerve of a category is a quasi-category, this functor induces a fully faithful functor from categories to quasi-categories. We will often consider this functor as an inclusion.

Definition 2.2 (Lurie)**.** The *underlying quasi-category* of a model category $\mathcal{M}$ is a quasi-category $\mathcal{U}(\mathcal{M})$ endowed with a functor $f\colon \mathcal{M} \to \mathcal{U}(\mathcal{M})$ such that for any quasi-category $\mathcal{D}$, the induced functor

$$f^*\colon\ \mathrm{Fun}(\mathcal{U}(\mathcal{M}), \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{M}, \mathcal{D})$$

is fully faithful with essential image the functors $\mathcal{M} \to \mathcal{D}$ which send weak equivalences of $\mathcal{M}$ to equivalences in $\mathcal{D}$. This quasi-category $\mathcal{U}(\mathcal{M})$, if it exists (and it does, see the next proposition), is determined uniquely up to equivalence of quasi-categories.

Remark 2.3*.* This definition differs slightly from the original definition of Lurie [Lur13, Definitions 1.3.4.1 and 1.3.4.15] in which one restricts to cofibrant objects of $\mathcal{M}$. Nevertheless, the two definitions are equivalent by [Lur13, Remark 1.3.4.16].

Remark 2.4*.* The definition of $\mathcal{U}(\mathcal{M})$ only depends on the underlying category of $\mathcal{M}$ and the weak equivalences of $\mathcal{M}$. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are two model categories on the same underlying category with same weak equivalences, then $\mathcal{U}(\mathcal{M})$ and $\mathcal{U}(\mathcal{N})$ are canonically equivalent.

Proposition 2.5 (Lurie)**.** *Every model category has an underlying quasi-category.*

Proof. This follows from [Lur13, Remark 1.3.4.2]. □

2.6. We will denote by $N_\Delta\colon \mathcal{C}\mathrm{at}_\Delta \to \mathcal{S}\mathrm{Set}$ Cordier's coherent nerve functor from simplicial categories to simplicial sets (see [Lur09a, Definition 1.1.5.5]). If $\mathcal{M}$ is a simplicial model category, we will denote by $\mathcal{M}^\circ$ the full simplicial subcategory of $\mathcal{M}$ spanned by the cofibrant and fibrant objects. This simplicial category $\mathcal{M}^\circ$ is locally fibrant in the sense that all its mapping spaces are Kan complexes. It follows from [CP86, Theorem 2.1] that its coherent nerve $N_\Delta(\mathcal{M}^\circ)$ is a quasi-category.

Theorem 2.7 (Lurie)**.** *If $\mathcal{M}$ is a simplicial model category, then $N_\Delta(\mathcal{M}^\circ)$ is the underlying quasi-category of $\mathcal{M}$.*

Proof. This is [Lur13, Theorem 1.3.4.20]. □

2.8. Recall from [Lur09a, Section 5.5] that the classical notion of locally presentable category can be generalized to the notion of locally presentable quasi-category.

By [Lur09a, Proposition A.3.7.6] and [Lur13, Proposition 1.3.4.22], these quasi-categories can be characterized as those being the underlying quasi-category of a combinatorial model category.

2.9. Denote by $\mathcal{S}$ the quasi-category of spaces, that is, the underlying quasi-category of the Kan–Quillen model structure on simplicial sets. If A is a small category, then the *quasi-category of presheaves* $\mathcal{P}(A)$ on A is the quasi-category $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$.

Proposition 2.10 (Heller, Bousfield–Kan)**.** *Let A be a small category. We have the following two simplicial proper combinatorial model structures on the category* $\mathrm{sPr}(A)$ *of simplicial presheaves on A:*

(i) *the* injective model structure*, whose weak equivalences and cofibrations are the objectwise weak equivalences and the objectwise cofibrations, respectively;*
(ii) *the* projective model structure*, whose weak equivalences and fibrations are the objectwise weak equivalences and the objectwise fibrations, respectively.*

Proof. It seems that the first appearances of the injective and projective model structures are [Hel88, Theorem 4.5] and [BK72, Chapter XI, §8], respectively. See [Lur09a, Proposition A.2.8.2] for a more general statement. The fact that these model structures are simplicial follows easily from the fact that the Kan–Quillen model structure on simplicial sets is simplicial. The left properness is obvious for the injective model structure and right properness for the projective model structure follows easily from right properness of the Kan–Quillen model structure. Since left and right properness only depend on the class of weak equivalences, it follows that these structures are both proper. □

We will denote by $\mathrm{sPr}(A)_{\mathrm{inj}}$ and $\mathrm{sPr}(A)_{\mathrm{proj}}$ these two model structures.

Proposition 2.11 (Lurie)**.** *The quasi-category $\mathcal{P}(A)$ of presheaves on a small category A is canonically equivalent to the underlying quasi-category of the projective model structure on* $\mathrm{sPr}(A)$*.*

Proof. This is a special case of [Lur13, Proposition 1.3.4.25]. □

2.12. If $\mathcal{C}$ is a quasi-category and X, Y are two objects of $\mathcal{C}$, we will denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the space of morphisms from X to Y. See [Lur09a, Section 1.2.2] or [DS11] for various approaches to define this space. Recall from [Lur09a, Section 2.2] that if $\mathcal{M}$ is a simplicial model category, then the mapping spaces of the underlying quasi-category of $\mathcal{M}$ can be computed using the simplicial enrichment of $\mathcal{M}$ (when restricted to cofibrant fibrant objects).

2.13. Let $\mathcal{C}$ be a quasi-category and let S be a class of morphisms of $\mathcal{C}$. An object Y of $\mathcal{C}$ is *S-local* if for every map $f\colon X \to X'$ in S, the induced map

$$f^*\colon \mathrm{Map}_{\mathcal{C}}(X', Y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak equivalence. The full subcategory of the quasi-category $\mathcal{C}$ spanned by the S-local objects is called the *localization of* $\mathcal{C}$ *by* S. We will denote it by $S^{-1}\mathcal{C}$.

Proposition 2.14 (Lurie)**.** *If $\mathcal{C}$ is a locally presentable quasi-category and S is a* set *of morphisms of $\mathcal{C}$, then the inclusion $i\colon S^{-1}\mathcal{C} \hookrightarrow \mathcal{C}$ admits a left adjoint L. In other words, we have a reflective localization*

$$L\colon \mathcal{C} \rightleftarrows S^{-1}\mathcal{C}\colon i.$$

Proof. This is [Lur09a, Proposition 5.5.4.15.(3)]. □

Remark 2.15. When the quasi-category $\mathcal{C}$ is an ordinary category A, the mapping space $\mathrm{Map}_A(X, Y)$ is simply the discrete simplicial set $A(X, Y)$. In particular, an object Y in A is S-local if and only if, for all $f\colon X \to X'$ in S, the induced map

$$f^*\colon A(X', Y) \longrightarrow A(X, Y)$$

is a bijection, or, in other words, if and only if Y is right orthogonal to S.

Proposition 2.16. *Let $\mathcal{M}$ be a left proper combinatorial model category and let S be a set of maps of $\mathcal{M}$. There is a canonical equivalence of quasi-categories*

$$S^{-1}(\mathcal{U}(\mathcal{M})) \xrightarrow{\simeq} \mathcal{U}(S^{-1}\mathcal{M}).$$

Proof. The functor $\mathcal{U}(\mathcal{M}) \to \mathcal{U}(S^{-1}\mathcal{M})$ sends S to equivalences and hence, by the universal property of the localization, we obtain a functor

$$\phi\colon S^{-1}(\mathcal{U}(\mathcal{M})) \longrightarrow \mathcal{U}(S^{-1}\mathcal{M}).$$

Let us show that this functor is an equivalence.

We first assume that $\mathcal{M}$ is a *simplicial* left proper combinatorial model category. By Theorem 2.7, the underlying quasi-categories of $\mathcal{M}$ and $S^{-1}\mathcal{M}$ are $N_\Delta(\mathcal{M}^\circ)$ and $N_\Delta((S^{-1}\mathcal{M})^\circ)$, respectively. The two quasi-categories $S^{-1}N_\Delta(\mathcal{M}^\circ)$ and $N_\Delta((S^{-1}\mathcal{M})^\circ)$ sit fully faithfully in $N_\Delta(\mathcal{M}^\circ)$. Moreover, using the compatibility between the mapping spaces of the model category $\mathcal{M}$ and those of the quasi-category $N_\Delta(\mathcal{M}^\circ)$, it is easy to check that these two quasi-categories are equal as subcategories of $N_\Delta(\mathcal{M}^\circ)$. The identity functor $S^{-1}N_\Delta(\mathcal{M}^\circ) \to N_\Delta((S^{-1}\mathcal{M})^\circ)$ is easily seen to be canonically equivalent to the functor ϕ, thereby concluding the proof of the simplicial case.

We now want to drop the additional assumption that $\mathcal{M}$ is simplicial by an application of a well-known result of Dugger. For that purpose, let us assume that we have a Quillen equivalence $\mathcal{M}_1 \rightleftarrows \mathcal{M}_2$ between left proper combinatorial model categories and a set S_2 of maps of $\mathcal{M}_2$. Let S_1 be the derived image of S_2 under the right adjoint of the Quillen pair. By [Hir03, Theorem 3.3.20], there is an induced Quillen equivalence $(S_1)^{-1}\mathcal{M}_1 \rightleftarrows (S_2)^{-1}\mathcal{M}_2$. Recall that Quillen equivalences between combinatorial model categories induce equivalences between the underlying quasi-categories [Lur13, Lemma 1.3.4.21]. We thus get a commutative diagram of quasi-categories

$$\begin{tikzcd}
\mathcal{U}(\mathcal{M}_1) \arrow[r] \arrow[d, "\simeq"'] & (S_1)^{-1}\mathcal{U}(\mathcal{M}_1) \arrow[r, "\phi_1"] \arrow[d, "\simeq"'] & \mathcal{U}((S_1)^{-1}\mathcal{M}_1) \arrow[d, "\simeq"] \\
\mathcal{U}(\mathcal{M}_2) \arrow[r] & (S_2)^{-1}\mathcal{U}(\mathcal{M}_2) \arrow[r, "\phi_2"'] & \mathcal{U}((S_2)^{-1}\mathcal{M}_2),
\end{tikzcd}$$

in which the vertical maps are equivalences. This implies that ϕ_1 is an equivalence if and only if ϕ_2 is an equivalence.

Finally, let $\mathcal{M}$ be a left proper combinatorial model category. Then by [Dug01], there is a left Quillen equivalence $\mathcal{N} \to \mathcal{M}$ such that $\mathcal{N}$ is a simplicial left proper combinatorial model category. The statement now follows easily from the previous two paragraphs. □

2.17. Let $\mathcal{C}$ be a quasi-category. An object Y of $\mathcal{C}$ is said to be 0-*truncated* if for every object X of $\mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X,Y)$ is discrete. We will denote by $\tau_{\leq 0}\mathcal{C}$ the full subcategory of $\mathcal{C}$ spanned by the 0-truncated objects of $\mathcal{C}$.

Proposition 2.18 (Lurie)**.** *If $\mathcal{C}$ is a locally presentable quasi-category, then the inclusion $i\colon \tau_{\leq 0}\mathcal{C} \hookrightarrow \mathcal{C}$ admits a left adjoint L. In other words, we have a reflective localization*

$$L\colon \mathcal{C} \rightleftarrows \tau_{\leq 0}\mathcal{C}\colon i.$$

Proof. This is [Lur09a, Proposition 5.5.6.18]. □

3. Autoequivalences and dense functors

In this section, we formalize the general strategy described in the introduction for computing certain quasi-categories of autoequivalences. This is made precise by Proposition 3.8 and will be used in the following three sections to determine the autoequivalences of (∞,n)-categories, ∞-operads and non-symmetric ∞-operads.

3.1. Let A be a small category. By Proposition 2.11, the quasi-category $\mathcal{P}(A)$ is canonically equivalent to $N_\Delta(\mathrm{sPr}(A)^{\mathrm{o}}_{\mathrm{proj}})$. Let us choose a functorial cofibrant replacement functor Q for $\mathrm{sPr}(A)_{\mathrm{proj}}$ with a natural trivial fibration $\phi\colon Q \to 1$. Since every discrete presheaf is fibrant in $\mathrm{sPr}(A)_{\mathrm{proj}}$, applying the functor Q to such a presheaf yields a cofibrant fibrant object of $\mathrm{sPr}(A)_{\mathrm{proj}}$. Thus the functor Q induces a morphism of simplicial categories $\mathrm{Pr}(A) \to \mathrm{sPr}(A)^{\mathrm{o}}_{\mathrm{proj}}$ and hence a morphism of quasi-categories $\mathrm{Pr}(A) \to \mathcal{P}(A)$.

Proposition 3.2. *Let A be a small category. Then the functor $\mathrm{Pr}(A) \to \mathcal{P}(A)$ is fully faithful, factors through the subcategory $\tau_{\leq 0}\mathcal{P}(A)$ of 0-truncated objects, and the restricted functor $\mathrm{Pr}(A) \to \tau_{\leq 0}\mathcal{P}(A)$ is an equivalence of quasi-categories.*

Proof. We begin by showing that the morphism $\mathrm{Pr}(A) \to \mathcal{P}(A)$ is fully faithful. Denote by $\delta\colon \mathrm{Pr}(A) \to \mathrm{sPr}(A)$ the functor sending a presheaf to the associated discrete simplicial presheaf. We have to show that if X and Y are two presheaves on A, then the map

$$Q\circ\delta\colon\ \mathrm{Map}_{\mathrm{Pr}(A)}(X,Y) \longrightarrow \mathrm{Map}_{\mathrm{sPr}(A)}(Q\delta X, Q\delta Y)$$

is a weak equivalence. Since the functor δ is fully faithful, this amounts to saying that the map

$$\mathrm{Map}_{\mathrm{sPr}(A)}(\delta X,\delta Y) \longrightarrow \mathrm{Map}_{\mathrm{sPr}(A)}(Q\delta X, Q\delta Y)$$

induced by Q is a weak equivalence. But this map sits in the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{sPr}(A)}(\delta X,\delta Y) \arrow[r, "\simeq"] \arrow[d, "Q"'] & \mathrm{Map}_{\mathrm{Pr}(A)}(\pi_0\delta X, Y) \arrow[dd, "\pi_0(\phi_{\delta X})^*"] \\
\mathrm{Map}_{\mathrm{sPr}(A)}(Q\delta X, Q\delta Y) \arrow[d, "(\phi_{\delta Y})_*"'] & \\
\mathrm{Map}_{\mathrm{sPr}(A)}(Q\delta X, \delta Y) \arrow[r, "\simeq"'] & \mathrm{Map}_{\mathrm{Pr}(A)}(\pi_0 Q\delta X, Y),
\end{tikzcd}$$

where the top and the bottom maps are isomorphisms coming from the adjunction (π_0,δ). Since $Q\delta X$ is cofibrant and $Q\delta Y \to \delta Y$ is a trivial fibration, the map $(\phi_{\delta Y})_*$ is a weak equivalence. The map $\pi_0(\phi_{\delta X})^*$ is even an isomorphism. It follows that the functor of the statement is fully faithful.

The fully faithfulness of the functor $\mathrm{Pr}(A) \to \mathcal{P}(A)$ easily implies that it factors through 0-truncated objects and it remains to show that the induced functor is essentially surjective. Given X in $\tau_{\leq 0}\mathcal{P}(A)$, we claim that the canonical map $\eta\colon X \to \delta\pi_0(X)$ is a natural equivalence. Obviously, η induces an isomorphism on π_0. To conclude, it thus suffices to show that for every object a in A, the space X_a is discrete. But we have $\mathrm{Map}_{\mathrm{sPr}(A)}(a, X) \simeq X_a$, where a denotes the discrete simplicial presheaf associated to a. This shows the result since X is 0-truncated. $\square$

Remark 3.3. It follows from the previous proposition that if S is a set of morphisms of $\mathrm{Pr}(A)$, then an object of $\mathrm{Pr}(A)$ is S-local (i.e., right orthogonal with respect to S) if and only if it is S-local when considered as an object of $\mathcal{P}(A)$.

Proposition 3.4. *Let $\mathcal{C}$ be a locally presentable quasi-category and let S be a set of morphisms between 0-truncated objects of $\mathcal{C}$. Then the quasi-categories $S^{-1}(\tau_{\leq 0}\mathcal{C})$ and $\tau_{\leq 0}(S^{-1}\mathcal{C})$ are equal as subcategories of $\mathcal{C}$.*

Proof. We have the following compositions of reflective localizations

$$\mathcal{C} \rightleftarrows S^{-1}\mathcal{C} \rightleftarrows \tau_{\leq 0}(S^{-1}\mathcal{C}), \qquad \mathcal{C} \rightleftarrows \tau_{\leq 0}\mathcal{C} \rightleftarrows S^{-1}(\tau_{\leq 0}\mathcal{C}),$$

and it hence suffices to show that the two quasi-categories $\tau_{\leq 0}(S^{-1}\mathcal{C})$ and $S^{-1}(\tau_{\leq 0}\mathcal{C})$ have the same objects. But if Y is an object of $S^{-1}(\tau_{\leq 0}\mathcal{C})$, then $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is discrete for all X in $\mathcal{C}$. In particular, $\mathrm{Map}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{Map}_{\mathcal{C}}(X, Y)$ is discrete for all X in $S^{-1}\mathcal{C}$ and hence Y lies in $\tau_{\leq 0}(S^{-1}\mathcal{C})$. Conversely, if Y is an object of $\tau_{\leq 0}(S^{-1}\mathcal{C})$, then $\mathrm{Map}_{S^{-1}\mathcal{C}}(X, Y)$ is discrete for all X in $S^{-1}\mathcal{C}$. Given an arbitrary X in $\mathcal{C}$, then, using the localization functor $L\colon \mathcal{C} \to S^{-1}\mathcal{C}$ and the S-locality of Y, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is weakly equivalent to the discrete space $\mathrm{Map}_{S^{-1}\mathcal{C}}(LX, Y)$, and so Y lies in $S^{-1}(\tau_{\leq 0}\mathcal{C})$. $\square$

Corollary 3.5. *Let A be a small category and let S be a set of morphisms of* $\mathrm{Pr}(A)$. *The morphism* $\mathrm{Pr}(A) \to \mathcal{P}(A)$ *induces an equivalence from the category* $S^{-1}\,\mathrm{Pr}(A)$ *to the quasi-category* $\tau_{\leq 0}(S^{-1}\mathcal{P}(A))$.

Proof. By Proposition 3.2, the morphism $\mathrm{Pr}(A) \to \mathcal{P}(A)$ induces an equivalence $\mathrm{Pr}(A) \to \tau_{\leq 0}\mathcal{P}(A)$ and hence an equivalence $S^{-1}\,\mathrm{Pr}(A) \to S^{-1}(\tau_{\leq 0}\mathcal{P}(A))$. But by the previous proposition, $S^{-1}(\tau_{\leq 0}\mathcal{P}(A))$ is nothing but $\tau_{\leq 0}(S^{-1}\mathcal{P}(A))$. $\square$

3.6. We will say that a functor $f\colon \mathcal{C} \to \mathcal{D}$ between quasi-categories is *dense* if the identity transformation $f \to f$ exhibits $1\colon \mathcal{D} \to \mathcal{D}$ as a left Kan extension of f along f. (In Lurie's terminology, see [Lur09b, Definition 4.4.2], one says that f strongly generates $\mathcal{D}$.) When $\mathcal{C}$ is small and $\mathcal{D}$ is cocomplete, this amounts to saying that the associated nerve functor (that is, the right adjoint to the canonical functor $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$) is fully faithful [Lur09b, Remark 4.4.4]. In particular, if A is a small category, then the Yoneda embedding $A \to \mathcal{P}(A)$ is dense. More generally, if A is a small category and S is a set of morphisms of $\mathcal{P}(A)$, then the canonical functor $A \to S^{-1}\mathcal{P}(A)$ is dense.

Proposition 3.7 (Lurie)**.** *Let $i\colon \mathcal{A} \to \mathcal{B}$ be a dense inclusion of quasi-categories such that $\mathcal{A}$ is small and $\mathcal{B}$ admits small colimits. If any autoequivalence of $\mathcal{B}$ restricts to an autoequivalence of $\mathcal{A}$, then the inclusion i induces a fully faithful functor $i^*\colon$ $\mathrm{Aut}(\mathcal{B}) \to \mathrm{Aut}(\mathcal{A})$.*

Proof. Denote by $\mathrm{Fun}^L(-,-)$ the quasi-category of colimit-preserving functors between two quasi-categories. The dense inclusion i induces a fully faithful functor $i^*\colon \mathrm{Fun}^L(\mathcal{B},\mathcal{B}) \to \mathrm{Fun}(\mathcal{A},\mathcal{B})$ by [Lur09b, Remark 4.4.5]. Consider the diagram

$$\begin{tikzcd} \mathrm{Fun}^L(\mathcal{B},\mathcal{B}) \arrow[r, "i^*"] & \mathrm{Fun}(\mathcal{A},\mathcal{B}) \\ \mathrm{Aut}(\mathcal{B}) \arrow[u] \arrow[ur] \arrow[r, dotted, "i^*"'] & \mathrm{Aut}(\mathcal{A}), \arrow[u] \end{tikzcd}$$

where the left vertical map is the canonical inclusion and the right vertical map is the fully faithful map given by postcomposing with i. By our assumption on autoequivalences of $\mathcal{B}$, the diagonal map factors over $\mathrm{Aut}(\mathcal{A})$, giving rise to the dotted morphism. Since all remaining maps are fully faithful, the same is true for $i^*\colon \mathrm{Aut}(\mathcal{B}) \to \mathrm{Aut}(\mathcal{A})$. $\square$

Proposition 3.8. *Let A be a small category and let S be a set of morphisms of $\mathrm{Pr}(A)$. Assume that the following conditions are satisfied:*

(i) *Representable presheaves in $\mathrm{Pr}(A)$ are S-local.*
(ii) *Any autoequivalence of $S^{-1}\,\mathrm{Pr}(A)$ restricts to an autoequivalence of A (the category A being included in $S^{-1}\,\mathrm{Pr}(A)$ because of (i)).*

Then $A \to S^{-1}\mathcal{P}(A)$ induces a fully faithful functor $\mathrm{Aut}(S^{-1}\mathcal{P}(A)) \to \mathrm{Aut}(A)$. In particular, the quasi-category $\mathrm{Aut}(S^{-1}\mathcal{P}(A))$ is discrete.

Proof. By Corollary 3.5, we know that there is an equivalence of quasi-categories $\tau_{\leq 0}(S^{-1}\mathcal{P}(A)) \simeq S^{-1}\,\mathrm{Pr}(A)$. Since 0-truncated objects are stable under equivalences, we obtain a functor $\mathrm{Aut}(S^{-1}\mathcal{P}(A)) \to \mathrm{Aut}(S^{-1}\,\mathrm{Pr}(A))$. Using assumption (ii), we get a functor $\mathrm{Aut}(S^{-1}\,\mathrm{Pr}(A)) \to \mathrm{Aut}(A)$. Obviously, the composition

$$\mathrm{Aut}(S^{-1}\mathcal{P}(A)) \longrightarrow \mathrm{Aut}(S^{-1}\,\mathrm{Pr}(A)) \longrightarrow \mathrm{Aut}(A)$$

of these two functors is induced by $A \to S^{-1}\,\mathrm{Pr}(A) \to S^{-1}\mathcal{P}(A)$. But this functor is dense (see the end of paragraph 3.6) and the result thus follows from the previous proposition. $\square$

4. Autoequivalences of the $(\infty,1)$-category of (∞,n)-categories

The aim of this section is to show that the quasi-category of autoequivalences of the quasi-category of Θ_n-spaces, which is a model for (∞,n)-categories, is the discrete category $(\mathbf{Z}/2\mathbf{Z})^n$. This calculation is a consequence of two results of Barwick and Schommer-Pries [BSP13], namely the computation of the autoequivalences of Υ_n-spaces and the comparison of Υ_n-spaces and Θ_n-spaces. Here we give a direct proof of this fact using the general strategy outlined in the introduction and formalized by Proposition 3.8. In particular, we reduce to the computation of the autoequivalences of Joyal's category Θ_n.

Throughout the section, we fix n such that $0 \leq n \leq \infty$.

4.1. Preliminaries on strict n-categories and the category Θ_n.

4.1.1. We will denote by $\mathcal{C}\mathrm{at}$ the category of small categories and by n-$\mathcal{C}\mathrm{at}_{\mathrm{str}}$ the category of small strict n-categories. Recall that these categories are complete and cocomplete. We refer the reader to [Ara13] for details on strict n-categories compatible with the notation we will use in this section.

4.1.2. Let k be such that $0 \le k \le n$. A strict k-category can be considered as a strict n-category whose i-arrows are identities for $i > k$. This defines a fully faithful functor $k\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}} \hookrightarrow n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$ which we will consider as an inclusion. In particular, we will say that a strict n-category is a k-category if it is in the image of this functor.

This inclusion functor admits both a left adjoint and a right adjoint. In this paper, we will only consider the right adjoint. It sends a strict n-category C to the strict k-category $\mathrm{tr}_k(C)$ obtained by throwing out the i-arrows of C for $i > k$. This k-category $\mathrm{tr}_k(C)$ will be called the *k-truncation* of C.

4.1.3. The *k-disk* D_k, where $k \ge 0$, is the strict ∞-category corepresenting the functor "set of k-arrows" $\mathrm{Ar}_k \colon \infty\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}} \to \mathcal{S}\mathrm{et}$. The strict ∞-category D_k is actually a k-category. Here are pictures of (the underlying ∞-graphs without the identities of) D_k in low dimension:

$$\mathrm{D}_0 = \bullet\,, \quad \mathrm{D}_1 = \begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}\,, \quad D_2 = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \quad \text{and} \quad D_3 = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Rrightarrow\Downarrow"] & \bullet \end{tikzcd}.$$

For $k > 0$, we have two morphisms $\sigma, \tau \colon \mathrm{D}_{k-1} \to \mathrm{D}_k$ corepresenting the natural transformations source and target $\mathrm{Ar}_k \to \mathrm{Ar}_{k-1}$, respectively. Concretely, σ (respectively τ) sends the unique non-trivial $(k-1)$-arrow of D_{k-1} to the source (respectively to the target) of the unique non-trivial k-arrow of D_k.

For $k > k' \ge 0$, we will also denote by σ and τ the morphisms $\mathrm{D}_{k'} \to \mathrm{D}_k$ obtained by composing

$$\mathrm{D}_{k'} \xrightarrow{\sigma} \mathrm{D}_{k'+1} \cdots \mathrm{D}_{k-1} \xrightarrow{\sigma} \mathrm{D}_k \quad \text{and} \quad \mathrm{D}_{k'} \xrightarrow{\tau} \mathrm{D}_{k'+1} \cdots \mathrm{D}_{k-1} \xrightarrow{\tau} \mathrm{D}_k,$$

respectively. Note that $\sigma, \tau \colon \mathrm{D}_{k'} \to \mathrm{D}_k$ are the only monomorphisms $\mathrm{D}_{k'} \hookrightarrow \mathrm{D}_k$.

4.1.4. A *table of dimensions* is a table

$$\begin{pmatrix} k_1 & & k_2 & & \cdots & & k_m \\ & k'_1 & & k'_2 & \cdots & k'_{m-1} & \end{pmatrix},$$

where $m \ge 1$, filled with non-negative integers satisfying $k_i > k'_i$ and $k_{i+1} > k'_i$ for every i such that $0 < i < m$. The greatest integer appearing in the table is called the *height* of the table.

To such a table T, we can associate the following diagram

$$\begin{tikzcd}[column sep=small] \mathrm{D}_{k_1} & & \mathrm{D}_{k_2} & & \mathrm{D}_{k_3} & \cdots & \mathrm{D}_{k_{m-1}} & & \mathrm{D}_{k_m} \\ & \mathrm{D}_{k'_1} \arrow[ul, "\sigma"] \arrow[ur, "\tau"'] & & \mathrm{D}_{k'_2} \arrow[ul, "\sigma"] \arrow[ur, "\tau"'] & & & & \mathrm{D}_{k'_{m-1}} \arrow[ul, "\sigma"] \arrow[ur, "\tau"'] & \end{tikzcd}$$

in $\infty\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$. We will denote this diagram by $\mathcal{D}_T$. The colimit of $\mathcal{D}_T$ will be denoted by $\Theta(T)$. We will also sometimes denote it by

$$\mathrm{D}_{k_1} \amalg_{\mathrm{D}_{k'_1}} \cdots \amalg_{\mathrm{D}_{k'_{m-1}}} \mathrm{D}_{k_m}.$$

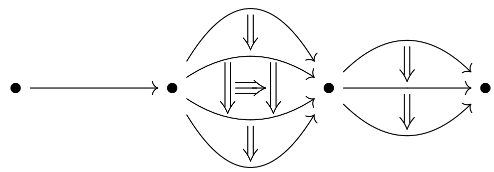
This is an n-category, where n is the height of T. Note that the diagram $\mathcal{D}_T$ actually comes from a diagram of n-graphs and it follows that the n-category $\Theta(T)$ is the free strict n-category on an n-graph $\Theta_0(T)$.

It is easy to see that the strict ∞-category $\Theta(T)$ does not admit any non-trivial automorphisms. This means, in particular, that there is a unique cocone making $\Theta(T)$ the colimit of the diagram $\mathcal{D}_T$.

Example 4.1.5. If T is the following table of dimensions

$$\begin{pmatrix} 2 & & 2 & & 2 & & 3 & & 2 & & 1 \\ & 1 & & 0 & & 1 & & 1 & & 0 & \end{pmatrix},$$

then $\Theta_0(T)$ is the following 3-graph:



and the 3-category $\Theta(T)$ associated to T is the strict 3-category freely generated by this 3-graph.

4.1.6. Joyal's category Θ_n of n-cells is defined in the following way. The objects of Θ_n are the tables of dimensions of height at most n. If S and T are two objects of Θ_n, then the set of morphisms from S to T is given by

$$\Theta_n(S, T) = n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}(\Theta(S), \Theta(T)).$$

By definition, we have a fully faithful functor $\Theta_n \to n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$. It is easy to see that this functor is also injective on objects and we will consider it as an inclusion. In particular, we will not make a difference between the table T and the associated strict ∞-category $\Theta(T)$.

Remark 4.1.7. Here are some comments on Θ_n for $n = 0, 1, \infty$:

(i) The category Θ_0 is the terminal category. It is *not* the category Θ_0 introduced by Berger in [Ber02], which corresponds to $\Theta_{\infty,0}$ with the notation of our paragraph 4.1.9.
(ii) The category Θ_1 is canonically isomorphic to the category Δ of simplices.
(iii) The category Θ_∞ is canonically isomorphic to the category Θ introduced by Joyal in [Joy97] by combinatorial means.

Remark 4.1.8. The category Θ_n has a universal property relating it to strict n-categories. Roughly speaking, it is the free category (having certain colimits) endowed with a strict n-cocategory object. See Propositions 3.11 and 3.14 of [Ara12] for the case $n = \infty$.

4.1.9. We define a category $\Theta_{n,0}$ in the following way: the objects of $\Theta_{n,0}$ are the same as the ones of Θ_n and the set of morphisms in $\Theta_{n,0}$ from an object S to an object T is given by

$$\Theta_{n,0}(S, T) = n\text{-}\mathcal{G}\mathrm{raph}(\Theta_0(S), \Theta_0(T)),$$

where n-$\mathcal{G}\mathrm{raph}$ denotes the category of n-graphs. The free strict n-category functor induces a canonical functor from $\Theta_{n,0}$ to Θ_n which is obviously faithful.

We will say that a morphism of Θ_n is *inert* if it comes from a morphism of $\Theta_{n,0}$. It is easy to see that inert morphisms are monomorphisms. A morphism $S \to T$ of Θ_n is said to be *active* if it does not factor through any non-trivial inert morphism $T' \to T$, that is, if every time it can be written as if, where i is inert, then i is an identity.

Active morphisms can be described more concretely in the following way. Let $f \colon \mathrm{D}_k \to T$ be a morphism of Θ_n. Such a morphism corresponds to a k-arrow of $\Theta(T)$. This k-arrow can be expressed using compositions and identities from some generators of the free strict n-category $\Theta(T)$, that is, from the arrows of $\Theta_0(T)$. It is not hard to see that the morphism f is active if and only if all the generators of $\Theta(T)$ are needed to express this k-arrow. In other words, $\mathrm{D}_k \to T$ is active if and only if it corresponds to (an identity in dimension k of) the total composition of $\Theta(T)$. This means that such an active morphism exists if and only if k is greater than or equal to the height of T, and that in this case, it is unique.

More generally, a morphism $S \to T$ of Θ_n corresponds to a pasting scheme of shape $\Theta_0(S)$ in $\Theta(T)$ and it is active if and only if all the generators of $\Theta(T)$ are needed to express the arrows of this pasting scheme.

Proposition 4.1.10 (Berger, Weber). *Every morphism of Θ_n can be written in a unique way as a composition of an active morphism followed by an inert morphism.*

Proof. This follows from the general machinery of [Web07] (see Example 4.21). This was first proved in [Ber02] (see Lemma 1.11) using a different but equivalent definition of Θ_n. See also [Ara10, Proposition 3.3.10]. □

4.1.11. The category of *n-cellular sets* is the category $\mathrm{Pr}(\Theta_n)$ of presheaves on Θ_n. The inclusion $\Theta_n \hookrightarrow n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$ induces a functor $N_n \colon n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}} \to \mathrm{Pr}(\Theta_n)$ sending a strict n-category C to the n-cellular set $T \mapsto n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}(T, C)$. This functor N_n is called the *n-cellular nerve.*

4.1.12. Let

$$T = \begin{pmatrix} k_1 & & k_2 & & \cdots & & k_m \\ & k'_1 & & k'_2 & \cdots & k'_{m-1} & \end{pmatrix}$$

be an object of Θ_n. By definition, we have

$$T = \mathrm{D}_{k_1} \amalg_{\mathrm{D}_{k'_1}} \cdots \amalg_{\mathrm{D}_{k'_{m-1}}} \mathrm{D}_{k_m}$$

in Θ_n. The *spine* I_T of T is the n-cellular set

$$I_T = \mathrm{D}_{k_1} \amalg_{\mathrm{D}_{k'_1}} \cdots \amalg_{\mathrm{D}_{k'_{m-1}}} \mathrm{D}_{k_m},$$

where the colimit is now taken in the category $\mathrm{Pr}(\Theta_n)$. There is a canonical morphism

$$i_T \colon I_T \longrightarrow T.$$

It is not hard to check that this morphism is a monomorphism. We will denote by $\mathcal{I}$ the set

$$\mathcal{I} = \{ i_T \mid T \in \mathrm{Ob}(\Theta_n) \}$$

of spine inclusions.

Let now X be an n-cellular set. For any object T of Θ_n, the map i_T induces a *Segal map*

$$X(T) \simeq \mathrm{Pr}(\Theta_n)(T, X) \longrightarrow \mathrm{Pr}(\Theta_n)(I_T, X) \simeq X_{k_1} \times_{X_{k'_1}} \cdots \times_{X_{k'_{m-1}}} X_{k_m},$$

where X_l means $X(\mathrm{D}_l)$. We will say that X *satisfies the Segal condition* if all the Segal maps are bijections. This exactly means that X is $\mathcal{I}$-local.

Proposition 4.1.13 (Berger, Weber). *The n-cellular nerve functor is fully faithful. Moreover, its essential image consists of the n-cellular sets satisfying the Segal condition.*

Proof. This follows from the general machinery of [Web07] (see Example 4.24). This was first proved in [Ber02] (see Theorem 1.12). □

Remark 4.1.14. The first assertion of the previous proposition means precisely that the inclusion functor $\Theta_n \hookrightarrow n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$ is dense.

4.2. The quasi-category of Θ_n-spaces.

4.2.1. The category of *n-cellular spaces* is the category $\mathrm{sPr}(\Theta_n) \simeq \mathrm{Pr}(\Theta_n \times \Delta)$ of simplicial presheaves on Θ_n. The first projection $p\colon \Theta_n \times \Delta \to \Theta_n$ induces a fully faithful functor $p^*\colon \mathrm{Pr}(\Theta_n) \to \mathrm{Pr}(\Theta_n \times \Delta)$ sending an n-cellular set to the corresponding discrete n-cellular space. We will always consider n-cellular sets as a full subcategory of n-cellular spaces using this functor.

4.2.2. Let $k \geq 1$. We will denote by J_k the strict ∞-category corepresenting the functor $\mathrm{Ar}_k^{\sim}\colon \infty\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}} \to \mathcal{S}\mathrm{et}$ sending a strict ∞-category to its set of strictly invertible k-arrows. The ∞-category J_k is actually a k-category. Here are pictures of (the underlying ∞-graphs without the identities of) J_k in low dimension:

$$J_1 = \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}, \quad J_2 = \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[from=U, to=D, phantom, "\Downarrow\Uparrow"] \end{tikzcd} \quad \text{and} \quad J_3 = \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[from=U, to=D, phantom, "\Downarrow \substack{\Rrightarrow \\ \Lleftarrow} \Downarrow"] \end{tikzcd}.$$

There is a canonical morphism $j_k\colon J_k \to \mathrm{D}_{k-1}$ corepresenting the natural transformation $\mathrm{Ar}_{k-1} \to \mathrm{Ar}_k^{\sim}$ sending a $(k-1)$-arrow to its identity. Concretely, the morphism j_k is the unique morphism $J_k \to \mathrm{D}_{k-1}$ sending the two non-trivial k-arrows of J_k to the identity of the only non-trivial $(k-1)$-arrow of D_{k-1}.

4.2.3. Recall from paragraph 4.1.12 that we denote by $\mathcal{I}$ the set
$$\mathcal{I} = \{i_T \mid T \in \mathrm{Ob}(\Theta_n)\}$$
of spine inclusions.

We will denote by $\mathcal{J}^\flat$ and $\mathcal{J}$ the sets
$$\mathcal{J}^\flat = \{j_k \mid 1 \leq k \leq n\} \quad \text{and} \quad \mathcal{J} = \{N_n(j_k) \mid 1 \leq k \leq n\}.$$
The sets $\mathcal{I}$ and $\mathcal{J}$ will be considered as sets of maps of n-cellular sets or n-cellular spaces depending on the context.

For the next two definitions and the proposition that follows, we will assume that n is finite.

Definition 4.2.4 (Rezk)**.** The *model category for Θ_n-spaces* is the left Bousfield localization of the injective model structure on n-cellular spaces by the set $\mathcal{I} \cup \mathcal{J}$. We will denote this model structure by $\mathrm{sPr}(\Theta_n)_{\Theta_n\text{-}\mathcal{S}p}$.

Remark 4.2.5. The original definition of Θ_n-spaces in [Rez10a] proceeds by induction on n. Here we are following the (trivially equivalent) definition given in [Ara14, Section 7].

Definition 4.2.6. The *quasi-category of Θ_n-spaces* is the localization of the quasi-category $\mathcal{P}(\Theta_n)$ by the set $\mathcal{I} \cup \mathcal{J}$. We will denote it by $\Theta_n\text{-}\mathcal{S}p$.

Proposition 4.2.7. *The quasi-category underlying the model category of Θ_n-spaces is canonically equivalent to the quasi-category of Θ_n-spaces.*

Proof. We indeed have
$$\mathcal{U}(\mathrm{sPr}(\Theta_n)_{\Theta_n\text{-}\mathcal{S}p}) = \mathcal{U}((\mathcal{I} \cup \mathcal{J})^{-1}\,\mathrm{sPr}(\Theta_n)_{\mathrm{inj}})$$

$$
\begin{aligned}
&\simeq (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{U}(\mathrm{sPr}(\Theta_n)_{\mathrm{inj}}) \\
&\quad \text{(by Proposition 2.16)} \\
&\simeq (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{U}(\mathrm{sPr}(\Theta_n)_{\mathrm{proj}}) \\
&\quad \text{(since } \mathcal{U} \text{ only depends on the weak equivalences)} \\
&\simeq (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{P}(\Theta_n) \\
&\quad \text{(by Proposition 2.11)} \\
&= \Theta_n\text{-}\mathcal{S}p.
\end{aligned}
$$

□

4.3. Autoequivalences of the category of strict n-categories.

4.3.1. Recall that for every i such that $1 \leq i \leq n$, there exists an autoequivalence op_i of the category $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$ sending a strict n-category C to the strict n-category obtained from C by reversing the orientation of the i-arrows. The functor op_i is obviously its own inverse. The mapping $i \mapsto \mathrm{op}_i$ extends formally to a monoid morphism

$$(\mathbf{Z}/2\mathbf{Z})^n \longrightarrow \mathrm{aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}),$$

where for $n = \infty$, we set $(\mathbf{Z}/2\mathbf{Z})^\infty = \prod_{k\geq 1} \mathbf{Z}/2\mathbf{Z}$. For $\delta = (\delta_i)_{1\leq i\leq n}$ an element of $(\mathbf{Z}/2\mathbf{Z})^n$, we will denote by op_δ the associated autoequivalence of $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$. Concretely, if C is a strict n-category, then $\mathrm{op}_\delta(C)$ is the strict n-category obtained from C by reversing the orientation of the i-arrows for all i such that $\delta_i = 1$.

Note that the monoid morphism $(\mathbf{Z}/2\mathbf{Z})^n \to \mathrm{aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}})$ defines a strict monoidal functor

$$(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}} \longrightarrow \mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}),$$

where $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$ denotes the discrete category on the set $(\mathbf{Z}/2\mathbf{Z})^n$ endowed with the strict monoidal structure given by the group law of $(\mathbf{Z}/2\mathbf{Z})^n$, and where the category $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}})$ is endowed with the strict monoidal structure given by composition of functors.

Remark 4.3.2. If T is an object of Θ_n, then it is easy to see that for any δ in $(\mathbf{Z}/2\mathbf{Z})^n$, we have $\mathrm{op}_\delta(T) \simeq T$.

4.3.3. Let $k \geq 0$. The *k-sphere* S^k is the k-truncation of the $(k+1)$-disk D_{k+1} (say in $\infty\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$). By definition, we have a canonical monomorphism $\mathrm{S}^k \hookrightarrow \mathrm{D}_{k+1}$. It is easy to prove by induction, setting $\mathrm{S}^{-1} = \varnothing$, that

$$\mathrm{S}^k \simeq \mathrm{D}_k \amalg_{\mathrm{S}^{k-1}} \mathrm{D}_k.$$

Lemma 4.3.4. *Let k be an integer such that $0 < k \leq n$. The k-disk D_k is the unique strict n-category C satisfying the following three properties:*

(i) *The n-category C is not a $(k-1)$-category.*
(ii) *Any proper sub-n-category of C is a $(k-1)$-category.*
(iii) *The $(k-1)$-truncation $\mathrm{tr}_{k-1}(C)$ of C is isomorphic to S^{k-1}.*

Proof. It is obvious that D_k satisfies these properties. Suppose C is a strict n-category satisfying these three properties. The first property exactly means that C has at least one non-trivial arrow in dimension at least k. Every such non-trivial arrow of C defines a sub-n-category which is not a $(k-1)$-category. The second property implies that this sub-n-category has to be C. This means that there can exist only one non-trivial arrow in C in dimension at least k. Let l be the dimension of this arrow. By the third property, the $(k-1)$-truncation of C is S^{k-1}.

Since the n-category generated by the unique non-trivial l-arrow of C is C, the two non-trivial $(k-1)$-arrows of S^{k-1} have to be iterated sources or targets of this unique non-trivial l-arrow. This means that l has to be equal to k and that the n-category C has to be the k-disk D_k. □

Lemma 4.3.5. *Let k be such that $0 \le k \le n$. A strict n-category C is a k-category if and only if there exists an extremal epimorphism of the form*

$$\coprod_E \mathrm{D}_k \longrightarrow C,$$

for some set E, that is, if there exists an epimorphism of the above form that does not factor through any proper subobject of C.

Proof. Clearly, such an extremal epimorphism exists if and only if C is generated by its set of k-arrows, that is, if and only if C is a k-category. □

4.3.6. Let F be an autoequivalence of n-$\mathcal{C}\mathrm{at}_{\mathrm{str}}$. Suppose that for some l such that $0 < l \le n$, we have

$$F(\mathrm{D}_{l-1}) \simeq \mathrm{D}_{l-1} \quad \text{and} \quad F(\mathrm{D}_l) \simeq \mathrm{D}_l.$$

Since the only monomorphisms from D_{l-1} to D_l are σ and τ, this implies that the equivalence F either fixes these two morphisms or exchanges them. We define $\delta_l(F)$ in $\mathbf{Z}/2\mathbf{Z}$ to be 0 if F fixes them and 1 otherwise.

Fix now k such that $0 < k \le n$ and suppose that we have $F(\mathrm{D}_l) \simeq \mathrm{D}_l$ for every l such that $0 \le l \le k$. We define $\delta_{\le k}(F)$ to be the element

$$\delta_{\le k}(F) = (\delta_1(F), \dots, \delta_k(F), 0, \dots, 0)$$

of $(\mathbf{Z}/2\mathbf{Z})^n$. In the case $k = n$, that is, in the case where F preserves all the disks, we set

$$\delta(F) = \delta_{\le n}(F) = (\delta_1(F), \dots, \delta_n(F)).$$

Proposition 4.3.7. *Every autoequivalence F of n-$\mathcal{C}\mathrm{at}_{\mathrm{str}}$ preserves the disks. In other words, we have $F(\mathrm{D}_k) \simeq \mathrm{D}_k$ for $0 \le k \le n$.*

Proof. We are going to prove by induction on k that F preserves both D_k and S^{k-1}. For $k = 0$, the objects D_0 and S^{-1} are terminal and initial objects of n-$\mathcal{C}\mathrm{at}_{\mathrm{str}}$, respectively, and hence they are preserved by F.

Suppose the result is true for $l < k$ and let us prove it for k. The equivalence F respects monomorphisms and reflects isomorphisms. It thus respects proper subobjects. It also reflects them since a quasi-inverse of F respects them. Moreover, since F preserves sums, extremal epimorphisms and the $(k-1)$-disk by induction, it preserves $(k-1)$-categories by Lemma 4.3.5. This shows that F preserves the two first conditions of Lemma 4.3.4.

We will now check that F preserves S^{k-1}. There are exactly two monomorphisms $\mathrm{S}^{k-2} \hookrightarrow \mathrm{D}_{k-1}$, say i_1 and i_2. By induction, F preserves S^{k-2} and D_{k-1}. It thus either fixes i_1 and i_2 or exchanges them. But for any value of ε in $\{1, 2\}$, we have a pushout square

$$\begin{tikzcd} \mathrm{S}^{k-2} \arrow[r, "i_\varepsilon"] \arrow[d, "i_\varepsilon"'] & \mathrm{D}_{k-1} \arrow[d] \\ \mathrm{D}_{k-1} \arrow[r] & \mathrm{S}^{k-1}. \end{tikzcd}$$

Since F preserves pushouts, we deduce that S^{k-1} is preserved.

Let us now prove that F is in some sense compatible with $(k-1)$-truncation. First, note that for any strict n-category C, the $(k-1)$-categories $\mathrm{tr}_{k-1}(F(C))$ and $F(\mathrm{tr}_{k-1}(C))$ (recall that we proved that F preserves $(k-1)$-categories) have the same arrows. Indeed, the set of i-arrows for $i<k$ is corepresented by D_i, which is preserved by F by induction. Since by induction F preserves D_l for l such that $0 \le l \le k-1$, we have by paragraph 4.3.6 an element $\delta_{\le k-1}(F)$ in $(\mathbf{Z}/2\mathbf{Z})^n$. It is immediate that the underlying n-graphs of $\mathrm{tr}_{k-1}(F(C))$ and $\mathrm{op}_{\delta_{\le k-1}(F)}(\mathrm{tr}_{k-1}(C))$ are equal.

In particular, if C is such that $\mathrm{tr}_{k-1}(C)$ is isomorphic to S^{k-1}, we get that the underlying n-graph of $\mathrm{tr}_{k-1}(F(C))$ is the underlying n-graph of

$$\mathrm{op}_{\delta_{\le k-1}(F)}(\mathrm{tr}_{k-1}(C)) \simeq \mathrm{op}_{\delta_{\le k-1}(F)}(\mathrm{S}^{k-1}) \simeq \mathrm{S}^{k-1}.$$

But since there exists only one structure of strict n-category on the underlying n-graph of S^{k-1}, we get that $\mathrm{tr}_{k-1}(F(C))$ is isomorphic to S^{k-1}. This shows that F preserves the last condition of Lemma 4.3.4.

We thus have proved that the three properties characterizing D_k are stable under F and hence we get that F preserves D_k. □

Remark 4.3.8. The analogous statement for *rigid* strict n-categories appears as [BSP13, Lemma 4.5]. The proof, which is based on a different characterization of the n-disk, also adapts to the case of strict n-categories.

4.3.9. Let F be an autoequivalence of $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$. By the previous proposition, F preserves all the disks and we thus get by paragraph 4.3.6 an element $\delta(F)$ in $(\mathbf{Z}/2\mathbf{Z})^n$. We thus have a map

$$\mathrm{aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}) \longrightarrow (\mathbf{Z}/2\mathbf{Z})^n.$$

It is immediate that this map is a monoid morphism and a retraction of the map $(\mathbf{Z}/2\mathbf{Z})^n \to \mathrm{aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}})$. In other words, if F and G are autoequivalences of $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$, we have $\delta(GF) = \delta(G)\delta(F)$, and if δ is an element of $(\mathbf{Z}/2\mathbf{Z})^n$, we have $\delta(\mathrm{op}_\delta) = \delta$.

Proposition 4.3.10. *If F is an autoequivalence of $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$, then we have $F(T) \simeq T$ for every object T of Θ_n. In particular, F induces an autoequivalence of Θ_n.*

Proof. By postcomposing F with $\mathrm{op}_{\delta(F)^{-1}}$ (which preserves the objects of Θ_n, as we already observed), we can suppose that $\delta(F) = 1$. Let T be an object of Θ_n. The object T seen as a strict n-category is the colimit of the diagram $\mathcal{D}_T$ of paragraph 4.1.4. But since $\delta(F) = 1$, the functor F preserves the diagram $\mathcal{D}_T$ and since F commutes with colimits, we indeed have $F(T) \simeq T$. □

Corollary 4.3.11. *The dense inclusion $\Theta_n \hookrightarrow n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$ induces a fully faithful functor $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}) \to \mathrm{Aut}(\Theta_n)$.*

Proof. This is immediate from the previous proposition and Proposition 3.7. □

We will show in Section 4.5 that the monoidal category $\mathrm{Aut}(\Theta_n)$ is isomorphic to the discrete monoidal category $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$. As a corollary, we will obtain the following theorem:

Theorem. *The functor $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}} \to \mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}})$ is an equivalence of monoidal categories.*

4.4. Autoequivalences of the category of rigid strict n-categories.

4.4.1. A strict n-category C is said to be *rigid* if it contains no non-trivial isomorphisms, that is, if for any k such that $1 \le k \le n$, any strictly invertible k-arrow of C is the identity of a $(k-1)$-arrow. For a fixed k, this condition amounts to saying that C is local with respect to the map $j_k \colon J_k \to \mathrm{D}_{k-1}$ of paragraph 4.2.2. In particular, a strict n-category C is rigid if and only if it is $\mathcal{J}^\flat$-local, where $\mathcal{J}^\flat$ is the set defined in paragraph 4.2.3.

We will denote by $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}$ the full subcategory of the category of strict n-categories whose objects are the rigid strict n-categories.

4.4.2. Let δ be an element of $(\mathbf{Z}/2\mathbf{Z})^n$. It is clear that if C is a rigid strict n-category, then so is $\mathrm{op}_\delta(C)$. In other words, the autoequivalence op_δ induces an autoequivalence of the category $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}$. We will also denote this equivalence by op_δ. We thus get, as in the non-rigid case, a monoidal functor

$$(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}} \longrightarrow \mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}).$$

Proposition 4.4.3. *If F is an autoequivalence of $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}$, then $F(T) \simeq T$ for every object T of Θ_n. In particular, F induces an autoequivalence of Θ_n.*

Proof. The proof used for $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$ in the previous subsection adapts trivially. One only has to observe that the objects of Θ_n and the spheres are rigid n-categories. □

Corollary 4.4.4. *The dense inclusion $\Theta_n \hookrightarrow n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}$ induces a fully faithful functor $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}) \to \mathrm{Aut}(\Theta_n)$.*

Proof. This is immediate from the previous proposition and Proposition 3.7. □

We will show in Section 4.5 that the monoidal category $\mathrm{Aut}(\Theta_n)$ is isomorphic to the discrete monoidal category $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$. As a corollary, we will obtain the following theorem, first proved by Barwick and Schommer-Pries [BSP13, Section 4] by different methods:

Theorem. *The functor $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}} \to \mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}})$ is an equivalence of monoidal categories.*

4.5. Autoequivalences of the category Θ_n.

4.5.1. Let δ be an element of $(\mathbf{Z}/2\mathbf{Z})^n$. As already observed, if T is an object of Θ_n, then we have $\mathrm{op}_\delta(T) \simeq T$. This means that the autoequivalence op_δ induces an autoequivalence of the category Θ_n. We will still denote this equivalence by op_δ. We thus get, as in the previous subsections, a monoidal functor

$$(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}} \longrightarrow \mathrm{Aut}(\Theta_n).$$

Lemma 4.5.2. *Let k be such that $0 < k \le n$. The k-disk D_k is the unique object T of Θ_n satisfying the following two properties:*

(i) *Every proper subobject of T is an l-disk for $l < k$.*
(ii) *For every l such that $0 \le l < k$, there are exactly two monomorphisms from D_l to T.*

Proof. Let

$$T = \begin{pmatrix} k_1 & & k_2 & & \cdots & & k_m \\ & k'_1 & & k'_2 & \cdots & k'_{m-1} & \end{pmatrix}$$

be an object of Θ_n satisfying the two properties of the statement. First note that we must have $m \leq 2$ for otherwise

$$\begin{pmatrix} k_1 & & k_2 & & \cdots & & & k_{m-1} \\ & k'_1 & & k'_2 & \cdots & k'_{m-2} & & \end{pmatrix}$$

would be a proper subobject that is not a disk, contradicting property (i). Suppose $m = 2$ so that

$$T = \begin{pmatrix} k_1 & & k_2 \\ & k'_1 & \end{pmatrix}.$$

There are exactly three monomorphisms from $\mathrm{D}_{k'_1}$ to such a T. Property (i) implies that $k'_1 < k$ and we get a contradiction with property (ii). This means that $m = 1$ or, in other words, that T is a disk D_p. Property (i) implies that $p \leq k$ and property (ii) that $p \geq k$, thereby proving the result. □

Proposition 4.5.3. *Every autoequivalence of Θ_n is the identity on objects.*

Proof. The strategy is similar to the one used for n-$\mathcal{C}\mathrm{at}_{\mathrm{str}}$ but has to be adapted since the spheres are not objects of Θ_n. Nevertheless, for k such that $0 < k \leq n$, the characterization of the k-disk given by Lemma 4.3.4 can be replaced by the one given by the above lemma. Using this characterization, we get by induction on k that D_k is preserved (starting from D_0 which is the terminal object of Θ_n). We then obtain that every object is preserved by expressing an object T of Θ_n as the colimit of the diagram $\mathcal{D}_T$ (using the same argument as in the proof of Proposition 4.3.10). □

Proposition 4.5.4. *The monoid morphism $(\mathbf{Z}/2\mathbf{Z})^n \to \mathrm{aut}(\Theta_n)$ is an isomorphism.*

Proof. It suffices to show that the retraction $F \mapsto \delta(F)$ (see paragraph 4.3.9) of the morphism of the statement is injective, that is, that if F is an autoequivalence of Θ_n such that $\delta(F) = 1$, then F is the identity.

Let us fix such an F. We know that F is the identity on objects by the previous proposition. Since morphisms of strict n-categories are determined by their action on arrows, it suffices to show that morphisms of the form $\mathrm{D}_k \to T$, where k is such that $0 \leq k \leq n$ and T is any object of Θ_n, are preserved.

Let $\mathrm{D}_k \to T$ be such a morphism. By Proposition 4.1.10, this morphism factors as a composite

$$\mathrm{D}_k \xrightarrow{a} S \xrightarrow{i} T,$$

where a is active and i is inert. Let us first prove that i is preserved by F. Recall that S and T are colimits of diagrams $\mathcal{D}_S$ and $\mathcal{D}_T$ involving disks only. Consider the cocone $\mathcal{D}_S \to T$ associated to i. Since i is inert, each of the components $\mathrm{D}_l \to T$ of this cocone corresponds to an l-cell of the n-graph $\Theta_0(T)$ generating $\Theta(T)$ and can thus be written (in a non-canonical way) as a composite

$$\mathrm{D}_l \longrightarrow \mathrm{D}_{l'} \longrightarrow \operatorname{colim} \mathcal{D}_T = T,$$

where the first map is either σ or τ, and the second map is one of the canonical morphisms associated to $\mathcal{D}_T$. Since $\delta(F) = 1$, the morphism $\mathrm{D}_l \to \mathrm{D}_{l'}$, the diagram $\mathcal{D}_S$ and the diagram $\mathcal{D}_T$ are preserved by F. Since moreover, as already observed, the cocone making T a colimit of the diagram $\mathcal{D}_T$ is unique, the canonical morphism $\mathrm{D}_{l'} \to T$ is also preserved. This shows that i is preserved.

Finally, let us prove that $a\colon \mathrm{D}_k \to S$ is preserved by F. Since there is at most one active morphism from a fixed disk to a fixed object of Θ_n (see paragraph 4.1.9), it suffices to show that $F(a)$ is active. This follows from the fact, just proved, that any autoequivalence of Θ_n preserves inert morphisms. Indeed, by Proposition 4.1.10, $F(a)$ factors uniquely as ib, where i is inert and b is active. This means that $a = F^{-1}(i)F^{-1}(b)$. By decomposing $F^{-1}(b)$, we get that $a = F^{-1}(i)jc$, where j is inert and c is active. Since F^{-1} preserves inert morphisms, $F^{-1}(i)$ is also inert. By uniqueness of the decomposition of a, we obtain that $F^{-1}(i)j$ is an identity. Since inert morphisms are monomorphisms and Θ_n has no non-trivial isomorphisms, this implies that $F^{-1}(i)$ and hence i are identities, and therefore that $F(a)$ is active. $\square$

Theorem 4.5.5. *The functor* $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}} \to \mathrm{Aut}(\Theta_n)$ *is an isomorphism of monoidal categories.*

Proof. The previous proposition states that this functor is bijective on objects. To conclude, it suffices to show that $\mathrm{Aut}(\Theta_n)$ is a discrete category. Let δ and δ' be two elements of $(\mathbf{Z}/2\mathbf{Z})^n$, and let $\gamma\colon \mathrm{op}_\delta \to \mathrm{op}_{\delta'}$ be a natural transformation. For every object T of Θ_n, we thus have an endomorphism γ_T of T.

Let us show by induction that for every $0 \le k \le n$, we have $\delta_k = \delta'_k$ and $\gamma_{\mathrm{D}_k} = 1_{\mathrm{D}_k}$. (A priori, δ_0 and δ'_0 are not defined and we define them to be both equal to 0 for the purpose of starting our induction.) The case $k = 0$ is obvious. For $k \ge 1$, consider the naturality squares associated to $\sigma, \tau\colon \mathrm{D}_{k-1} \to \mathrm{D}_k$:

$$
\begin{tikzcd}
\mathrm{D}_{k-1} \arrow[r, "\mathrm{op}_\delta(\sigma)"] \arrow[d, "\gamma_{\mathrm{D}_{k-1}}=1_{\mathrm{D}_{k-1}}"'] & \mathrm{D}_k \arrow[d, "\gamma_{\mathrm{D}_k}"] \\
\mathrm{D}_{k-1} \arrow[r, "\mathrm{op}_{\delta'}(\sigma)"'] & \mathrm{D}_k,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{D}_{k-1} \arrow[r, "\mathrm{op}_\delta(\tau)"] \arrow[d, "\gamma_{\mathrm{D}_{k-1}}=1_{\mathrm{D}_{k-1}}"'] & \mathrm{D}_k \arrow[d, "\gamma_{\mathrm{D}_k}"] \\
\mathrm{D}_{k-1} \arrow[r, "\mathrm{op}_{\delta'}(\tau)"'] & \mathrm{D}_k.
\end{tikzcd}
$$

These squares determine the value of γ_{D_k} on the $(k-1)$-truncation S^{k-1} of D_k: it is one of the two automorphisms of S^{k-1}. But only the trivial automorphism can be lifted to an endomorphism of D_k and the unique lift is then the identity of D_k. This exactly means that δ_k and δ'_k are equal and that γ_{D_k} is the identity of D_k.

This shows that if such a γ exists, then $\delta = \delta'$. To conclude, we have to show that the identity is the unique natural transformation $\gamma\colon \mathrm{op}_\delta \to \mathrm{op}_\delta$. Fix an object T of Θ_n and consider the naturality squares

$$
\begin{tikzcd}
\mathrm{D}_k \arrow[r, "\mathrm{op}_\delta(u)"] \arrow[d, "1_{\mathrm{D}_k}"'] & T \arrow[d, "\gamma_T"] \\
\mathrm{D}_k \arrow[r, "\mathrm{op}_\delta(u)"'] & T
\end{tikzcd}
$$

associated to morphisms of the form $u\colon \mathrm{D}_k \to T$ where $0 \le k \le n$. Since a morphism of strict n-categories is determined by its action on arrows, there is at most one morphism γ_T making all these squares commute, namely the identity of T, thereby proving the result. $\square$

Theorem 4.5.6. *The monoidal categories* $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}})$ *and* $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}})$ *are equivalent to the discrete monoidal category* $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$.

Proof. By Corollaries 4.3.11 and 4.4.4, we have that the categories $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}})$ and $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}})$ are both full (monoidal) subcategories of the category $\mathrm{Aut}(\Theta_n)$. To conclude, it thus suffices to show that every autoequivalence of Θ_n lifts to autoequivalences of $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str}}$ and $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}$. This is obvious since, by Proposition 4.5.4, the autoequivalences of Θ_n are the op_δ. □

Remark 4.5.7. The fact that the category $\mathrm{Aut}(n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}})$ is equivalent to the discrete category $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$ was first obtained by Barwick and Schommer-Pries in [BSP13, Section 4].

4.6. Autoequivalences of the $(\infty,1)$-category of (∞,n)-categories.

In this section, we suppose that n is finite.

4.6.1. Let δ be an element of $(\mathbf{Z}/2\mathbf{Z})^n$. The autoequivalence op_δ of Θ_n extends formally to an autoequivalence of the quasi-category $\mathcal{P}(\Theta_n)$. It is easy to see that the sets $\mathcal{I}$ and $\mathcal{J}$ of paragraph 4.2.3 are stable under this autoequivalence and we thus get an induced autoequivalence op_δ of $\Theta_n\text{-}\mathcal{S}p = (\mathcal{I}\cup\mathcal{J})^{-1}\mathcal{P}(\Theta_n)$.

Proposition 4.6.2. *An n-cellular set is $(\mathcal{I}\cup\mathcal{J})$-local if and only if it is the nerve of a rigid strict n-category.*

Proof. Proposition 4.1.13 precisely says that an n-cellular set is $\mathcal{I}$-local if and only if it is the nerve of a strict n-category. By paragraph 4.4.1, such an n-cellular set is $\mathcal{J}$-local if and only if the strict n-category of which it is the nerve is rigid, thereby proving the result. □

Theorem 4.6.3. *The quasi-category* $\mathrm{Aut}(\Theta_n\text{-}\mathcal{S}p)$ *is canonically equivalent to the discrete category* $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$.

Proof. We are going to apply Proposition 3.8 to $A = \Theta_n$ and $S = \mathcal{I}\cup\mathcal{J}$. Let us check that the hypotheses are fulfilled. Using the previous proposition, this amounts to verifying that

(i) objects of Θ_n are rigid strict n-categories;
(ii) autoequivalences of $n\text{-}\mathcal{C}\mathrm{at}_{\mathrm{str,r}}$ restrict to autoequivalences of Θ_n.

The first point is obvious and the second point is Proposition 4.4.3. We can thus apply the proposition and we get that $\mathrm{Aut}(\Theta_n\text{-}\mathcal{S}p)$ is a full subcategory of $\mathrm{Aut}(\Theta_n)$. But $\mathrm{Aut}(\Theta_n)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$ by Theorem 4.5.5. To conclude, it thus suffices to show that every autoequivalence of Θ_n lifts to an autoequivalence of $\Theta_n\text{-}\mathcal{S}p$. This follows from paragraph 4.6.1. □

Remark 4.6.4. The above result is a consequence of previous work of Barwick and Schommer-Pries. Indeed, by [BSP13, Theorem 11.15] the quasi-category $\Theta_n\text{-}\mathcal{S}p$ is equivalent to the quasi-category of Υ_n-spaces, and by [BSP13, Theorem 8.12] the quasi-category of autoequivalences of Υ_n-spaces is $(\mathbf{Z}/2\mathbf{Z})^n$.

Remark 4.6.5. Let $\mathcal{C}$ be a quasi-category and let $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ be the associated quasi-category of endofunctors. Recall from [Lur07, Proposition 3.1.7] that $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ is a monoidal quasi-category, the monoidal structure being given by the composition of endofunctors (for the general theory of *non-symmetric* monoidal quasi-categories, see [Lur07]). Moreover, the full subcategory $\mathrm{Aut}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{C})$ spanned by the autoequivalences inherits a monoidal structure.

Using these monoidal structures, Theorem 4.6.3 admits the following refinement: the quasi-categories $\mathrm{Aut}(\Theta_n\text{-}\mathcal{S}p)$ and $(\mathbf{Z}/2\mathbf{Z})^n_{\mathrm{disc}}$ are equivalent as monoidal quasi-categories.

5. Autoequivalences of the $(\infty, 1)$-category of ∞-operads

In this section, we show that the quasi-category of autoequivalences of the quasi-category of Ω-spaces, which is a model for ∞-operads, is a contractible Kan complex. We follow the general strategy described in the introduction and formalized by Proposition 3.8. In particular, we reduce to the computation of the autoequivalences of the category Ω of trees.

5.1. Preliminaries on operads and the category of trees.

5.1.1. We will denote by $\mathcal{O}\mathrm{p}$ the category of (small) symmetric coloured operads (see for instance [EM06, Section 2] where they are called multicategories). We will refer to its objects simply as operads.

If P is an operad and $c_1, \ldots, c_n$ and d are colours of P, we will denote by $P(c_1, \ldots, c_n; d)$ the set of operations of P whose input colours are given by the n-tuple $(c_1, \ldots, c_n)$ and whose output colour is d. If $p \in P(c_1, \ldots, c_n; d)$ is an n-ary operation and σ is an element of the symmetric group Σ_n, we will denote by $p\sigma$ the induced operation in $P(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; d)$.

We will denote by $\mathcal{C}\mathrm{oll}$ the category of symmetric (coloured) collections and by $\mathcal{C}\mathrm{oll}_{\mathrm{ns}}$ the category of non-symmetric (coloured) collections. Recall that a symmetric collection K consists of a set of *colours* and, for every $(n+1)$-tuple of colours $(c_1, \ldots, c_n, d)$ with $n \geq 0$, a set $K(c_1, \ldots, c_n; d)$ with an action of Σ_n. Non-symmetric collections are described in the same way, but forgetting the action of the symmetric groups. We have forgetful functors

$$\mathcal{O}\mathrm{p} \longrightarrow \mathcal{C}\mathrm{oll} \longrightarrow \mathcal{C}\mathrm{oll}_{\mathrm{ns}}$$

and these functors admit left adjoints.

If p and q are two operations of an operad P, we will write $p \sim_\Sigma q$ if p and q have the same arity n and there exists an element σ in Σ_n such that $q = p\sigma$.

We will say that an operation of an operad is *non-trivial* if it is not the identity of a colour.

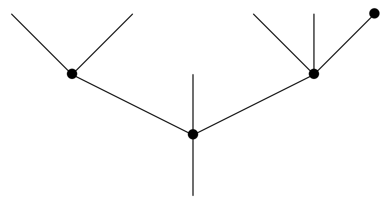
We will identify the category $\mathcal{C}\mathrm{at}$ of small categories with the full subcategory of $\mathcal{O}\mathrm{p}$ consisting of operads having only unary operations. The inclusion functor $\mathcal{C}\mathrm{at} \hookrightarrow \mathcal{O}\mathrm{p}$ admits a right adjoint sending an operad to its so-called *underlying category*. Concretely, the underlying category of an operad P is the suboperad of P obtained by throwing out the non-unary operations (in particular, its objects are the colours of P).

5.1.2. A *combinatorial tree* is a non-empty finite connected graph with no loops. A vertex of a combinatorial tree is said to be *outer* if it has only one edge attached to it.

An *operadic tree* is a combinatorial tree endowed with the choice of an outer vertex called the *output* and of a (possibly empty) set of outer vertices not containing the output called the set of *inputs*. The choice of the output induces an orientation of the tree "from the inputs to the output". A *vertex* of an operadic tree is defined as a vertex of the underlying combinatorial tree which is neither an input nor the output. An *edge* of an operadic tree is an edge of the underlying combinatorial tree.

The edge attached to the output will be called the *root* and the edges attached to the inputs will be called the *leaves*.

We will follow the usual conventions when drawing operadic trees: the output will be drawn at the bottom of the tree and the vertices corresponding to the inputs and the output will be deleted. Here is an example of such a tree:



From now on, by a *tree*, we will mean an operadic tree.

A *planar structure* on a tree T consists of the data of an ordering of the input edges of each vertex v of T. A *planar tree* is a tree endowed with a planar structure. We will sometimes use *non-planar tree* as a synonym of tree in order to emphasize that a given tree is not planar.

5.1.3. To every tree T, we can associate a symmetric collection $K(T)$. The colours of $K(T)$ are the edges of T and for every vertex v of T with n inputs and every choice of an order $e_1 < \cdots < e_n$ on the input edges of v, there is an operation in $K(T)(e_1, \ldots, e_n; e)$, where e is the output edge of v. The group Σ_n acts on the operations associated to v in the obvious way.

If T is a tree, we will denote by $\Omega(T)$ the free symmetric operad on the symmetric collection $K(T)$.

Similarly, to every planar tree $\overline{T}$, we can associate a non-symmetric collection $K_p(\overline{T})$. Its colours are the edges of $\overline{T}$ and for every vertex v in $\overline{T}$ with input edges $e_1 < \cdots < e_n$ and output edge e, there is an operation in $K_p(\overline{T})(e_1, \ldots, e_n; e)$.

Note that if T is the underlying non-planar tree of a planar tree $\overline{T}$, then $K(T)$ is the free symmetric collection on $K_p(\overline{T})$. (Another way to put this is to say that the choice of a planar structure on T corresponds to a choice of generators of $K(T)$.) In particular, $\Omega(T)$ is the free symmetric operad on $K_p(\overline{T})$.

5.1.4. The category of trees Ω, introduced by Moerdijk and Weiss in [MW07], is defined as follows: the objects of Ω are trees up to isomorphism (that is, up to renaming of their vertices and edges) and the set of morphisms in Ω from an object S to an object T is given by

$$\Omega(S, T) = \mathcal{O}\mathrm{p}(\Omega(S), \Omega(T)).$$

By definition, there is a canonical fully faithful functor $\Omega \hookrightarrow \mathcal{O}\mathrm{p}$ and we will always consider Ω as a full subcategory of $\mathcal{O}\mathrm{p}$ using this functor.

We will denote by η the tree with one edge and no vertices. For $n \geq 0$, we will denote by C_n the n-corolla, that is, the tree with one vertex and n leaves. Note that C_n, seen as an object of $\mathcal{O}\mathrm{p}$, corepresents the functor "set of n-ary operations". Similarly, η corepresents the functor "set of colours". In particular, for any tree T, we have a *root map* $\eta \to T$ and *leaf maps* $\eta \to T$.

We now fix for every object T of Ω the choice of a planar structure on T. We will denote the resulting planar tree by $\overline{T}$. The purpose of these choices is to make precise the idea that every tree can be obtained by glueing corollas (see Proposition 5.1.8).

5.1.5. Given a tree T, we will denote by $\Omega_0(T)$ the non-symmetric collection $K_p(\overline{T})$. We define a category Ω_0 in the following way: the objects of Ω_0 are the same as the ones of Ω and the set of morphisms in Ω_0 from an object S to an object T is given by

$$\Omega_0(S,T) = \mathcal{C}\mathrm{oll}_{\mathrm{ns}}(\Omega_0(S), \Omega_0(T)).$$

The free symmetric operad functor on a non-symmetric collection induces a canonical functor from Ω_0 to Ω which is obviously faithful. We will always consider Ω_0 as a subcategory of Ω using this functor.

Remark 5.1.6. The subcategory Ω_0 of Ω *does* depend on the choice of the planar structures. One way to avoid these choices is to replace Ω by the following equivalent category $\overline{\Omega}$: an object of $\overline{\Omega}$ is an object of Ω endowed with the choice of a planar structure and the set of morphisms in $\overline{\Omega}$ from an object $\overline{S}$ to an object $\overline{T}$ is given by

$$\overline{\Omega}(\overline{S},\overline{T}) = \mathcal{O}\mathrm{p}(\Omega(S), \Omega(T)),$$

where S and T denote the respective underlying non-planar trees of $\overline{S}$ and $\overline{T}$. It is then possible to define a canonical subcategory $\overline{\Omega}_0$ of $\overline{\Omega}$ by setting

$$\overline{\Omega}_0(\overline{S},\overline{T}) = \mathcal{C}\mathrm{oll}_{\mathrm{ns}}(K_p(\overline{S}), K_p(\overline{T})).$$

Note that the choice of planar structures we made corresponds to the choice of a section of the equivalence of categories given by the forgetful functor $\overline{\Omega} \to \Omega$.

5.1.7. We will denote by $\mathbb{C}$ the full subcategory of Ω_0 whose objects are η and the corollas. The morphisms of $\mathbb{C}$, besides the identities, are exactly the root and the leaf maps of corollas. We will denote by j the inclusion functor $\mathbb{C} \hookrightarrow \Omega_0$.

The category of non-symmetric collections can be identified with the category of presheaves on $\mathbb{C}$. Indeed, a presheaf F on $\mathbb{C}$ is given by a set $F(\eta)$ (corresponding to the set of colours) and, for every $n \geq 0$, a set $F(C_n)$ endowed with a morphism $F(C_n) \to F(\eta)^{n+1}$ induced by the n leaf maps and the root map $\eta \to C_n$. But such a map amounts to a family of sets indexed by $(n+1)$-tuples $(c_1, \dots, c_n, d)$ of elements of $F(\eta)$.

If T is an object of Ω, we will denote by $\mathbb{C}/T$ the comma category $j \downarrow T$, where T is seen as an object of Ω_0. An object of $\mathbb{C}/T$ is hence a pair $(C, C \to T)$, where C is an object of $\mathbb{C}$ and $C \to T$ is a morphism of Ω_0. A morphism from $(C, C \to T)$ to $(C', C' \to T)$ is a morphism $C \to C'$ in $\mathbb{C}$ (i.e., in Ω_0) making the obvious triangle commute. Thus, an object of $\mathbb{C}/T$ is just a colour or a generating operation of T.

We will denote by $\mathcal{D}_T$ the functor $\mathbb{C}/T \to \mathbb{C} \hookrightarrow \Omega$.

Proposition 5.1.8. *For every object T of Ω, the canonical morphism*

$$\operatorname{colim} \mathcal{D}_T = \operatorname*{colim}_{(C, C\to T)\in \mathbb{C}/T} C \longrightarrow T$$

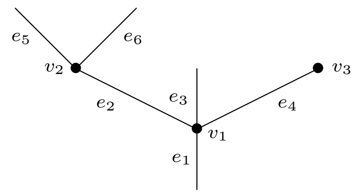
is an isomorphism of Ω. Moreover, the inclusion functor $\Omega \hookrightarrow \mathcal{O}\mathrm{p}$ preserves this colimit.

Proof. Since Ω is a full subcategory of $\mathcal{O}\mathrm{p}$, it suffices to prove that we have a canonical isomorphism when the colimit is taken in $\mathcal{O}\mathrm{p}$. The canonical decomposition of the non-symmetric collection $K_p(\overline{T})$, seen as a presheaf over $\mathbb{C}$, as a colimit of representable presheaves gives a canonical isomorphism of non-symmetric collections

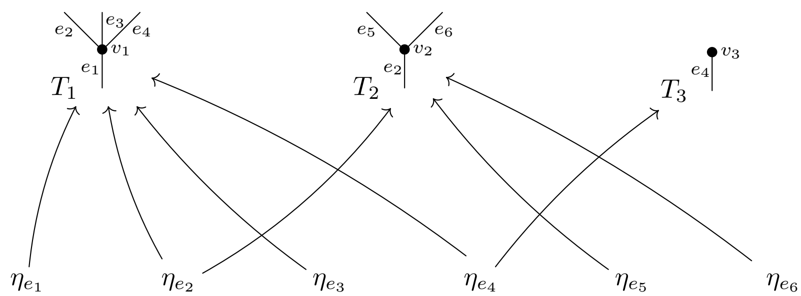
$$\operatorname*{colim}_{(C, C\to T)\in \mathbb{C}/T} K_p(C) \longrightarrow K_p(\overline{T}).$$

The result then follows from the fact that the free symmetric operad functor commutes with colimits. □

Example 5.1.9. Let T be the tree



endowed with the planar structure given by the picture. We define T_i as the corolla associated to the vertex v_i, i.e., T_i has the input edges of v_i as leaves and the output edge of v_i as root. Then, the diagram $\mathcal{D}_T$ has the corollas T_i and a copy η_e of η for every edge e in T as objects. The morphisms in the diagram are the morphisms from each η_e to the corresponding edge of T_i. Thus, the diagram $\mathcal{D}_T$ is the following:



5.1.10. The category of *dendroidal sets* is the category $\mathrm{Pr}(\Omega)$ of presheaves on Ω. The inclusion $\Omega \hookrightarrow \mathcal{O}\mathrm{p}$ induces a functor $N_d \colon \mathcal{O}\mathrm{p} \to \mathrm{Pr}(\Omega)$ sending an operad P to the dendroidal set $T \mapsto \mathcal{O}\mathrm{p}(T, P)$. This functor N_d is called the *dendroidal nerve*.

5.1.11. Let T be a tree. The *spine* of T is the dendroidal set

$$I_T = \operatorname*{colim}_{(C, C \to T) \in \mathbb{C}/T} C,$$

where the colimit is taken in $\mathrm{Pr}(\Omega)$. By Proposition 5.1.8, there is a canonical morphism of dendroidal sets $i_T \colon I_T \to T$. It is not hard to check that this morphism is a monomorphism. We will denote by $\mathcal{I}$ the set

$$\mathcal{I} = \{ i_T \mid T \in \mathrm{Ob}(\Omega) \}$$

of spine inclusions.

Let now X be a dendroidal set. For any tree T, the map i_T induces a *Segal map*

$$X(T) \simeq \mathrm{Pr}(\Omega)(T, X) \longrightarrow \mathrm{Pr}(\Omega)(I_T, X) \simeq \lim_{(C, C \to T) \in \mathbb{C}/T} X(C).$$

We will say that X *satisfies the Segal condition* if all the Segal maps are bijections. This exactly means that X is $\mathcal{I}$-local.

Proposition 5.1.12 (Cisinski–Moerdijk, Weber)**.** *The dendroidal nerve functor is fully faithful. Moreover, its essential image consists of the dendroidal sets satisfying the Segal condition.*

Proof. This is [CM13a, Corollary 2.6]. It also follows from the general machinery of [Web07] (see Example 4.27). □

Remark 5.1.13. The first assertion of the previous proposition precisely means that the inclusion functor $\Omega \hookrightarrow \mathcal{O}\mathrm{p}$ is dense.

5.2. The quasi-category of complete dendroidal Segal spaces.

5.2.1. The category of *dendroidal spaces* is the category $\mathrm{sPr}(\Omega) \simeq \mathrm{Pr}(\Omega \times \Delta)$ of simplicial presheaves on Ω. The first projection $p\colon \Omega \times \Delta \to \Omega$ induces a fully faithful functor $p^*\colon \mathrm{Pr}(\Omega) \to \mathrm{Pr}(\Omega \times \Delta)$ sending a dendroidal set to the corresponding discrete dendroidal space. We will always consider dendroidal sets as a full subcategory of dendroidal spaces using this functor.

5.2.2. Let X be a dendroidal set and let T be a tree. Then, by functoriality, the group $\mathrm{aut}_\Omega(T)$ acts on $X(T)$. More generally, if $f\colon X \to Y$ is a monomorphism of dendroidal sets, then $\mathrm{aut}_\Omega(T)$ acts on the difference $Y(T)\backslash f_T(X(T))$. Such a monomorphism is said to be *normal* if this action is free for any T.

More generally, a monomorphism of dendroidal spaces $f\colon X \to Y$ will be called *normal* if the monomorphism of dendroidal sets $f_{\bullet,n}\colon X_{\bullet,n} \to Y_{\bullet,n}$ is normal for every $n \geq 0$.

Proposition 5.2.3 (Cisinski–Moerdijk)**.** *There is a simplicial proper combinatorial model structure on* $\mathrm{sPr}(\Omega)$ *whose weak equivalences are the objectwise simplicial weak homotopy equivalences and whose cofibrations are the normal monomorphisms.*

Proof. The existence of this model structure is given by [CM13a, Proposition 5.2], as well as the fact that it is proper and combinatorial.

Let us prove it is simplicial. We have to show that the pushout product $f \,\square\, g$ of a monomorphism f of simplicial sets and a normal monomorphisms g of dendroidal spaces is again a normal monomorphisms of dendroidal sets, and that moreover, if either f or g is a weak equivalence, so is $f \,\square\, g$. Using the fact that pushouts in a presheaf category are computed objectwise, the first statement amounts to showing that the pushout product of a map of sets and a normal monomorphism of dendroidal sets is again a normal monomorphism of dendroidal sets. This follows at once by inspection. The second statement follows immediately from the fact that the Kan–Quillen model structure is cartesian closed. $\square$

Following [CM13a] and [BM11], the model structure of the previous proposition will be called the *generalized Reedy model structure* on dendroidal spaces. We will denote it by $\mathrm{sPr}(\Omega)_{\mathrm{Reedy}}$.

5.2.4. Recall that the category of operads is endowed with a closed symmetric monoidal structure given by the so-called Boardman–Vogt tensor product [BV73, Definition 2.14] (see also [MT10, Part I, Section 4.1]). We will denote this tensor product by $\otimes_{\mathcal{O}\mathrm{p}}$. The associated internal hom will be denoted by $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}$. The unit of this tensor product is the operad η. In particular, for any operad P, we have a canonical isomorphism $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(\eta, P) \simeq P$.

The only thing we will need to know about $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}$ is the description of its underlying category: if P and Q are two operads, then the set of objects of the underlying category of $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(P, Q)$ is the set of maps of operads from P to Q; if f, g are two such maps, a morphism from f to g is a natural transformation α from f to g, that is, the data of an operation α_c in $Q(f(c); g(c))$ for every colour c of P such that for every operation p in $P(c_1, \dots, c_n; d)$ we have

$$g(p) \circ (\alpha_{c_1}, \dots, \alpha_{c_n}) = \alpha_d \circ f(p) \in Q(f(c_1), \dots, f(c_n); g(d)).$$

5.2.5. Recall from paragraph 5.1.11 that we denote by $\mathcal{I}$ the set

$$\mathcal{I} = \{i_T \mid T \in \mathrm{Ob}(\Omega)\}$$

of spine inclusions.

Let J be the contractible groupoid on two objects and let $j\colon J \to \eta$ be the unique map of operads. For any tree T, by tensoring j with T we obtain an induced map of operads

$$j_T\colon J \otimes_{\mathcal{O}\mathrm{p}} T \longrightarrow \eta \otimes_{\mathcal{O}\mathrm{p}} T \xrightarrow{\simeq} T.$$

Note that j_η is canonically isomorphic to j. We will denote by $\mathcal{J}^\flat$ and $\mathcal{J}$ the sets

$$\mathcal{J}^\flat = \{j_T \mid T \in \mathrm{Ob}(\Omega)\} \quad \text{and} \quad \mathcal{J} = \{N_d(j_T) \mid T \in \mathrm{Ob}(\Omega)\}.$$

The sets $\mathcal{I}$ and $\mathcal{J}$ will be considered as sets of maps in dendroidal sets or dendroidal spaces depending on the context.

Definition 5.2.6 (Cisinski–Moerdijk)**.** The *model category for complete dendroidal Segal spaces* or *Ω-spaces* is the left Bousfield localization of the generalized Reedy model structure on dendroidal spaces by the set $\mathcal{I} \cup \mathcal{J}$. We will denote this model structure by $\mathrm{sPr}(\Omega)_{\Omega\text{-}\mathcal{S}p}$.

Remark 5.2.7. This definition is slightly different from the one appearing in [CM13a, Definition 6.2], which uses the tensor product of dendroidal sets. The two definitions are immediately seen to be equivalent using that $N_d(J \otimes_{\mathcal{O}\mathrm{p}} T) \simeq N_d(J) \otimes T$ (here $\otimes$ denotes the tensor product of dendroidal sets induced by the Boardman–Vogt tensor product [MW07, Section 5]); see [Wei07, Lemma 4.3.3].

Definition 5.2.8. The *quasi-category of Ω-spaces* is the localization of the quasi-category $\mathcal{P}(\Omega)$ by the set $\mathcal{I} \cup \mathcal{J}$. We will denote it by $\Omega\text{-}\mathcal{S}p$.

Proposition 5.2.9. *The quasi-category underlying the model category of Ω-spaces is canonically equivalent to the quasi-category of Ω-spaces.*

Proof. We indeed have

$$\begin{aligned}
\mathcal{U}(\mathrm{sPr}(\Omega)_{\Omega\text{-}\mathcal{S}p}) &= \mathcal{U}((\mathcal{I} \cup \mathcal{J})^{-1}\, \mathrm{sPr}(\Omega)_{\mathrm{Reedy}}) \\
&\simeq (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{U}(\mathrm{sPr}(\Omega)_{\mathrm{Reedy}}) \\
&\qquad \text{(by Proposition 2.16)} \\
&\simeq (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{U}(\mathrm{sPr}(\Omega)_{\mathrm{proj}}) \\
&\qquad \text{(since } \mathcal{U} \text{ only depends on the weak equivalences)} \\
&\simeq (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{P}(\Omega) \\
&\qquad \text{(by Proposition 2.11)} \\
&= \Omega\text{-}\mathcal{S}p.
\end{aligned}$$

□

5.3. Autoequivalences of the category of operads.

Lemma 5.3.1. *Let P be an operad. The following are equivalent:*

(i) *The operad P is a category.*
(ii) *There exists a morphism from P to η.*
(iii) *There exists exactly one morphism from P to η.*
(iv) *For every non-empty operad Q, there exists a morphism from P to Q.*

Proof. The equivalence of the three first properties is obvious. It is clear that (iv) implies (ii). Conversely, if P satisfies (ii) and Q is a non-empty operad, we obtain a map from P to Q by composing $P \to \eta \to Q$, where the second map corresponds to the choice of a colour of Q. □

Proposition 5.3.2. *Let F be an autoequivalence of $\mathcal{O}\mathrm{p}$. If C is a category, then $F(C)$ is a category. In particular, F restricts to an equivalence of $\mathcal{C}\mathrm{at}$.*

Proof. Since the empty operad is the initial object of $\mathcal{O}\mathrm{p}$, it has to be preserved by F. By applying this remark to a quasi-inverse of F, we immediately get that property (iv) of the previous lemma is preserved by F. The first assertion thus follows from the lemma. This shows that F induces an endofunctor of $\mathcal{C}\mathrm{at}$. This functor is immediately seen to be an equivalence of categories by applying the first assertion to a quasi-inverse of F. □

Corollary 5.3.3. *For every autoequivalence F of $\mathcal{O}\mathrm{p}$ we have $F(\eta) \simeq \eta$. In particular, if P is an operad, then there is a natural bijection between the sets of colours of P and $F(P)$.*

Proof. Since η is the terminal object of $\mathcal{C}\mathrm{at}$, the first assertion follows from the fact that F restricts to an equivalence of $\mathcal{C}\mathrm{at}$. The second assertion follows since η corepresents the functor "set of colours". □

Definition 5.3.4. An operad is said to be *discrete* if it can be written as a coproduct of copies of η. An operad is said to be *non-discrete* if it is not discrete, i.e., if it has at least one non-trivial operation.

Definition 5.3.5. A *pseudo-corolla* is an operad P having a non-trivial operation p satisfying the following two properties:

(i) For every non-trivial operation q of P we have $p \sim_\Sigma q$.
(ii) The only colours of P are the inputs and the output of p.

Remark 5.3.6. Roughly speaking, a pseudo-corolla is a corolla where the input and output colours do not have to be distinct. More precisely, a pseudo-corolla is a corolla if and only if the input and output colours of its unique non-trivial operation (up to permutation) are distinct.

Note that a pseudo-corolla is *not* determined by an arity and its colours. Indeed, there are two pseudo-corollas P with colours a, b and having an operation p in $P(a, a; b)$. One has no other operations and a trivial action of Σ_2 and the other one has another operation q in $P(a, a; b)$ and the transposition of Σ_2 acts by exchanging p and q.

Note also that if P is a pseudo-corolla, then any non-trivial operation of P satisfies conditions (i) and (ii) of the definition.

Remark 5.3.7. The purpose of the second condition of the definition is to exclude operads like $C_n \sqcup \eta$.

Lemma 5.3.8. *Let P be an operad. Then the following are equivalent:*

(i) *The operad P is a pseudo-corolla.*
(ii) *The operad P is non-discrete and every proper suboperad of P is discrete.*

Proof. Every pseudo-corolla satisfies (ii) by definition. Conversely, suppose P satisfies (ii). Since P is non-discrete, it has at least one non-trivial operation p. If

q is another non-trivial operation of P, then we must have $p \sim_\Sigma q$, for otherwise q would generate a non-discrete proper suboperad of P. If c is a colour of P, then c is necessarily an input or the output of p, for otherwise p would generate a non-discrete proper suboperad of P. This shows that P is a pseudo-corolla. $\square$

Proposition 5.3.9. *Every autoequivalence of* $\mathcal{O}\mathrm{p}$ *preserves pseudo-corollas.*

Proof. Let F be an autoequivalence of $\mathcal{O}\mathrm{p}$. Since F preserves η (Corollary 5.3.3) and coproducts, it also preserves discrete operads. Applying the same argument to a quasi-inverse of F, we get that F preserves non-discrete operads. It is then immediate to see that the characterization of pseudo-corollas given by the previous lemma is stable under F. $\square$

Lemma 5.3.10. *Let P be an operad. Then the following are equivalent:*

(i) *The operad P is the n-corolla.*
(ii) *The operad P is a pseudo-corolla satisfying the following property: for every pseudo-corolla Q, if there exists a map $Q \to P$, then Q has exactly $n+1$ colours.*

Proof. If $P = C_n$ is the n-corolla, then it is a pseudo-corolla and if there is a map f from a pseudo-corolla Q to C_n, then Q has to have exactly $n+1$ colours, since f sends operations of arity n to operations of arity n and is surjective on the colours.

Conversely, let P be a pseudo-corolla satisfying (ii). Note that taking $Q = P$ and the identity map, we get that P has exactly $n+1$ colours. If P has arity n, then $P = C_n$ and we are done. If P had arity $m \neq n$, then there would be a map from C_m to P. But C_m has $m+1$ colours and P would thus not satisfy (ii). $\square$

Proposition 5.3.11. *Every autoequivalence F of* $\mathcal{O}\mathrm{p}$ *preserves corollas. In particular, if P is an operad, then there is a natural bijection between the sets of n-ary operations of P and $F(P)$ for all n.*

Proof. Using the characterization of n-corollas given by the previous lemma, this immediately follows from the fact that F preserves pseudo-corollas and the number of colours (Proposition 5.3.9 and Lemma 5.3.10). $\square$

Proposition 5.3.12. *Let F be an autoequivalence of* $\mathcal{O}\mathrm{p}$*. Then, for every $n \geq 0$, F preserves the root map of the corolla C_n.*

Proof. Let $r\colon \eta \to C_n$ be the root map of C_n. By Corollary 5.3.3, $F(\eta)$ is isomorphic to η and hence corepresents colours. Let $r'\colon F(\eta) \to F(C_n)$ be the corresponding root map of $F(C_n)$. We have to prove that $F(r) = r'$. Recall moreover that by Proposition 5.3.11, $F(C_n)$ is an n-corolla.

The case $n = 0$ being obvious, let us assume $n \geq 1$. Suppose on the contrary that $F(r) \neq r'$. This means that $F(r)\colon F(\eta) \to F(C_n)$ is a leaf map (see paragraph 5.1.4). Since F is faithful, only one colour $\eta \to C_n$ can be sent to the root map. In particular, when $n = 2$, there exists a leaf map $d\colon \eta \to C_2$ which is not sent to the root map. Now let $C_2 \circ_d C_n$ be the pushout of the following diagram

$$\begin{tikzcd} \eta \arrow[r, "r"] \arrow[d, "d"'] & C_n \arrow[d, dotted] \\ C_2 \arrow[r, dotted] & C_2 \circ_d C_n. \end{tikzcd}$$

Thus, the operad $C_2 \circ_d C_n$ is the tree obtained by identifying the leaf d of C_2 with the root of C_n.

Since F preserves pushouts, we obtain a pushout diagram

$$\begin{tikzcd} F(\eta) \arrow[r, "F(r)"] \arrow[d, "F(d)"'] & F(C_n) \arrow[d] \\ F(C_2) \arrow[r] & F(C_2 \circ_d C_n). \end{tikzcd}$$

A description of this pushout (keeping in mind that $F(r)$ and $F(d)$ are leaf maps) reveals that $F(C_2 \circ_d C_n)$ has only non-trivial operations in arity 2 and n. If $n \geq 2$, this is a contradiction, since $C_2 \circ_d C_n$ has an operation of arity $n+1$ and F preserves this property by Proposition 5.3.11. Similarly, if $n = 1$, we get a contradiction since $C_2 \circ_d C_1$ has strictly more binary operations than $F(C_2 \circ_d C_1)$. □

5.3.13. Let F be an autoequivalence of $\mathcal{O}\mathrm{p}$ and let T be a tree. Recall from paragraph 5.1.7 that we have a diagram $\mathcal{D}_T\colon \mathbb{C}/T \to \Omega$. We will now consider it as a diagram in $\mathcal{O}\mathrm{p}$ using the inclusion functor $\Omega \hookrightarrow \mathcal{O}\mathrm{p}$. The purpose of this paragraph is to construct a canonical natural isomorphism $\phi\colon \mathcal{D}_T \to F\mathcal{D}_T$.

Let $(C, C \to T)$ be an object of $\mathbb{C}/T$. Suppose first $C = \eta$. By Corollary 5.3.3, we have $F(\eta) \simeq \eta$ and since η has no non-trivial endomorphism, there is a unique morphism $\eta \to F(\eta)$. We define $\phi_{(\eta,\eta\to T)}$ to be this unique morphism $\eta \to F(\eta)$.

Suppose now $T = C_n$ for $n \geq 0$. By Proposition 5.3.11, $F(C_n)$ is an n-corolla. Since F is fully faithful, it induces a bijection $\mathcal{O}\mathrm{p}(\eta, C_n) \to \mathcal{O}\mathrm{p}(F(\eta), F(C_n))$ or, in other words, a bijection between the colours of C_n and the colours $F(C_n)$. By Proposition 5.3.12, this bijection sends the root of C_n to the root of $F(C_n)$ and we hence get a bijection between the leaves of the n-corolla C_n and the leaves of the n-corolla $F(C_n)$. This bijection determines a unique morphism $C_n \to F(C_n)$ and we define $\phi_{(C_n,C_n\to T)}$ to be this morphism.

By definition, $\phi_{(C_n,C_n\to T)}$ is the unique morphism from C_n to $F(C_n)$ such that the square

$$\begin{tikzcd} \eta \arrow[r, "c"] \arrow[d, "\phi_{(\eta,\eta\to T)}"'] & C_n \arrow[d, "\phi_{(C_n,C_n\to T)}"] \\ F(\eta) \arrow[r, "F(c)"'] & F(C_n) \end{tikzcd}$$

commutes for every colour $c\colon \eta \to C_n$. This precisely means that ϕ is indeed a natural transformation and hence a natural isomorphism since its components are obviously isomorphisms.

Remark 5.3.14. The natural isomorphism ϕ is actually the unique natural transformation from $\mathcal{D}_T$ to $F\mathcal{D}_T$, as the components of ϕ are determined by the naturality squares.

Proposition 5.3.15. *If F is an autoequivalence of $\mathcal{O}\mathrm{p}$, then $F(T) \simeq T$ for every object T of Ω. In particular, F induces an autoequivalence of Ω.*

Proof. Using the canonical decomposition of T (Proposition 5.1.8), the canonical isomorphism $\mathcal{D}_T \simeq F\mathcal{D}_T$ of the previous paragraph and the fact that F commutes with colimits, we obtain a chain of canonical isomorphisms

$$T \simeq \operatorname{colim} \mathcal{D}_T \simeq \operatorname{colim} F\mathcal{D}_T \simeq F(\operatorname{colim} \mathcal{D}_T) \simeq F(T),$$

thereby proving the result. □

Corollary 5.3.16. *The dense inclusion* $\Omega \hookrightarrow \mathcal{O}\mathrm{p}$ *induces a fully faithful functor* $\mathrm{Aut}(\mathcal{O}\mathrm{p}) \to \mathrm{Aut}(\Omega)$.

Proof. This is immediate from the previous proposition and Proposition 3.7. □

We will show in Section 5.5 that the category $\mathrm{Aut}(\Omega)$ is a contractible groupoid. As a corollary, we will obtain the following theorem:

Theorem. *The category* $\mathrm{Aut}(\mathcal{O}\mathrm{p})$ *is a contractible groupoid.*

5.4. Autoequivalences of the category of rigid operads.

5.4.1. An operad is *rigid* if its underlying category is rigid, that is, if every invertible unary operation is the identity of a colour. In other words, an operad P is rigid if it is j-local, where $j\colon J \to \eta$ is the map of paragraph 5.2.5.

We will denote by $\mathcal{O}\mathrm{p}_{\mathrm{r}}$ the full subcategory of $\mathcal{O}\mathrm{p}$ whose objects are the rigid operads. Note that the operads induced by trees are rigid. We hence get a canonical inclusion functor $\Omega \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{r}}$. We also have a canonical inclusion functor $\mathcal{C}\mathrm{at}_{\mathrm{r}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{r}}$.

Proposition 5.4.2. *If F is an autoequivalence of* $\mathcal{O}\mathrm{p}_{\mathrm{r}}$*, then $F(T) \simeq T$ for every object T of Ω. In particular, F induces an autoequivalence of Ω.*

Proof. The strategy of the proof is similar to the one used for $\mathcal{O}\mathrm{p}$ in the previous subsection. Each of the results of that subsection has an obvious variant for $\mathcal{O}\mathrm{p}_{\mathrm{r}}$ obtained by inserting the adjective "rigid" at appropriate places. We leave it to the reader to check that the proofs of these results adapt trivially using the fact that the empty operad, operads induced by trees, pseudo-corollas, discrete operads and the operad $C_2 \circ_d C_n$ appearing in the proof of Proposition 5.3.12 are rigid. □

Corollary 5.4.3. *The dense inclusion* $\Omega \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{r}}$ *induces a fully faithful functor* $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{r}}) \to \mathrm{Aut}(\Omega)$.

Proof. This is immediate from the previous proposition and Proposition 3.7. □

We will show in Section 5.5 that the category $\mathrm{Aut}(\Omega)$ is a contractible groupoid. As a corollary, we will obtain the following theorem:

Theorem. *The category* $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{r}})$ *is a contractible groupoid.*

5.5. Autoequivalences of the category of trees.

5.5.1. Let A be a small category. Let us denote by Σ_A the group $\prod_{a \in \mathrm{Ob}(A)} \mathrm{aut}_A(a)$. To any element $\sigma = (\sigma_a)$ in Σ_A, we associate an endofunctor F_σ of A in the following way:

(i) For any object a in A, we set $F_\sigma(a) = a$.
(ii) For any morphism $f\colon a \to b$ in A, we set $F_\sigma(f) = \sigma_b f \sigma_a^{-1}$.

This assignment defines a monoid morphism $\Sigma_A \to \mathrm{aut}(A)$. In general, this morphism is neither injective nor surjective. Furthermore, for any σ and σ' in Σ_A, we have a canonical natural isomorphism from F_σ to $F_{\sigma'}$ whose a-th component is $\sigma'_a \sigma_a^{-1}$.

Let us denote by $(\Sigma_A)_{\mathrm{contr}}$ the contractible groupoid on Σ_A, that is, the category whose objects are the elements of Σ_A and with a unique morphism between any two objects. The groupoid $(\Sigma_A)_{\mathrm{contr}}$ is canonically endowed with the strict monoidal structure given by the group structure of Σ_A. By the previous paragraph, we have a canonical strict monoidal functor $(\Sigma_A)_{\mathrm{contr}} \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is endowed with the strict monoidal structure given by composition.

The purpose of this subsection is to show that for $A = \Omega$, the canonical functor $(\Sigma_\Omega)_{\mathrm{contr}} \to \mathrm{Aut}(\Omega)$ is an isomorphism of strict monoidal categories.

Proposition 5.5.2. *Every autoequivalence of Ω is the identity on objects.*

Proof. The strategy of the proof is similar to the one used for $\mathcal{O}\mathrm{p}$ and $\mathcal{O}\mathrm{p}_{\mathrm{r}}$ in the previous subsections. Note that Ω is a skeletal category and it thus suffices to show that $F(T) \simeq T$ for every tree T.

Lemma 5.3.1 can be adapted to Ω by saying that a tree T is linear if and only if, for every tree S, there exists a least one map from $S \to T$. This shows that F restricts to an equivalence of the full subcategory of Ω whose objects are linear trees (which is nothing but the category Δ). Since η is the terminal object of this category, it has to be preserved by F. In particular, for every tree T, the objects T and $F(T)$ have the same number of colours. Furthermore, the n-corolla can be characterized in Ω as the unique tree which only has η as a proper subobject and which has $n+1$ colours. It has therefore to be preserved by F.

The proof of Proposition 5.3.12 can be adapted to show that F also preserves root maps of corollas (one has to observe that a diagram $C_2 \leftarrow \eta \to C_n$, where the arrows are leaf maps, does not admit a pushout in Ω). If now T is an arbitrary tree, then using the canonical decomposition of T (Proposition 5.1.8) and the natural isomorphism $\phi\colon \mathcal{D}_T \to F\mathcal{D}_T$ of functors $\mathbb{C}/T \to \Omega$ constructed in paragraph 5.3.13, we get a chain of isomorphisms

$$T \simeq \operatorname{colim} \mathcal{D}_T \simeq \operatorname{colim} F\mathcal{D}_T \simeq F(\operatorname{colim} \mathcal{D}_T) \simeq F(T),$$

thereby proving the result. $\square$

5.5.3. Let F be an autoequivalence of Ω and let T be an object of Ω. Recall from paragraphs 5.1.7 and 5.3.13 that we have a diagram $\mathcal{D}_T\colon \mathbb{C}/T \to \Omega$ and a canonical natural isomorphism $\phi\colon \mathcal{D}_T \to F\mathcal{D}_T$. We will denote by $\sigma(F)_T$ the automorphism of T given by the composition of the canonical isomorphisms

$$T = \operatorname{colim} \mathcal{D}_T \longrightarrow \operatorname{colim} F\mathcal{D}_T \longrightarrow F(\operatorname{colim} \mathcal{D}_T) = F(T) = T.$$

Unravelling the definitions, this means that $\sigma(F)_T$ is the unique endomorphism of T such that for any object $(C, i\colon C \to T)$ of $\mathbb{C}/T$, we have

$$\sigma(F)_T \circ i = F(i) \circ \phi_{(C,i\colon C \to T)}.$$

Since maps of Ω are uniquely determined by their action on the colours, $\sigma(F)_T$ is uniquely determined by the following property: for every morphism $c\colon \eta \to T$ of Ω, we have

$$(\star) \qquad \sigma(F)_T \circ c = F(c).$$

(We are using here the equality $\phi_{(\eta, c\colon \eta \to T)} = 1_\eta$.) Note that by definition of ϕ, if $(C_n, C_n \to T)$ is an object of $\mathbb{C}/T$, we have $\phi_{(C_n, C_n \to T)} \circ c = F(c)$ and hence

$$\phi_{(C_n, C_n \to T)} = \sigma(F)_{C_n}.$$

We will denote by $\sigma(F)$ the element of Σ_Ω whose component at a tree T is given by $\sigma(F)_T$.

Lemma 5.5.4. *The assignment $F \mapsto \sigma(F)$ satisfies the following properties:*

(i) *For any σ in Σ_Ω, we have $\sigma(F_\sigma) = \sigma$.*
(ii) *For any autoequivalence F of Ω, we have $F = F_{\sigma(F)}$.*

Proof. Let σ be an element of Σ_Ω. To prove the first point, it suffices to check that for any colour $c\colon \eta \to T$ of Ω, we have $\sigma(F_\sigma)_T \circ c = \sigma_T \circ c$. But we have

$$\sigma(F_\sigma)_T \circ c = F_\sigma(c) = \sigma_T \circ c,$$

where the first equality holds by $(\star)$ and the second one by definition.

Let us prove the second point. By Proposition 5.5.2, F is the identity on objects. The same is true for $F_{\sigma(F)}$ by definition. Since maps of Ω are determined by their action on the colours, any autoequivalence of Ω which is the identity on objects is determined by its action on the maps whose source is η. We thus have to check that for any colour $c\colon \eta \to T$ of a tree T, we have $F_{\sigma(F)}(c) = F(c)$. This is indeed the case since

$$F_{\sigma(F)}(c) = \sigma(F)_T \circ c = F(c),$$

where the first equality holds by definition and the second one by $(\star)$. □

Proposition 5.5.5. *The monoid morphism $\Sigma_\Omega \to \mathrm{aut}(\Omega)$ is an isomorphism.*

Proof. The previous lemma precisely says that the map $F \mapsto \sigma(F)$ is an inverse to the map of the statement. □

Theorem 5.5.6. *The functor $(\Sigma_\Omega)_{\mathrm{contr}} \to \mathrm{Aut}(\Omega)$ is an isomorphism of categories. In particular, the category $\mathrm{Aut}(\Omega)$ is a contractible groupoid.*

Proof. The previous proposition states that this functor is bijective on objects. To conclude, it suffices to show that there exists a unique isomorphism between F_σ and $F_{\sigma'}$, where σ and σ' are two elements of Σ_Ω. The functor of the statement gives a map from F_σ to $F_{\sigma'}$. Let us prove its uniqueness. Let $\gamma\colon F_\sigma \to F_{\sigma'}$ be a natural transformation. For any tree T and any colour $c\colon \eta \to T$, the naturality square

$$\begin{tikzcd}
F_\sigma(\eta) = \eta \arrow[r, "F_\sigma(c)"] \arrow[d, "\gamma_\eta = 1_\eta"'] & F_\sigma(T) = T \arrow[d, "\gamma_T"] \\
F_{\sigma'}(\eta) = \eta \arrow[r, "F_{\sigma'}(c)"] & F_{\sigma'}(T) = T
\end{tikzcd}$$

shows that the action of γ_T on the colours is uniquely determined. The morphism γ_T is hence uniquely determined, thereby proving the result. □

Theorem 5.5.7. *The categories $\mathrm{Aut}(\mathcal{O}\mathrm{p})$ and $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{r}})$ are contractible groupoids.*

Proof. By Corollaries 5.3.16 and 5.4.3, the categories $\mathrm{Aut}(\mathcal{O}\mathrm{p})$ and $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{r}})$ are both full subcategories of the category $\mathrm{Aut}(\Omega)$. The result then follows immediately from the previous theorem, since full subcategories of contractible groupoids are again contractible groupoids. □

5.6. Autoequivalences of the quasi-category of ∞-operads.

Proposition 5.6.1. *Let P be an operad. Then the following are equivalent:*

(i) *The operad P is rigid.*
(ii) *For any tree T, the operad $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(T,P)$ is rigid.*
(iii) *For any operad Q, the operad $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(Q,P)$ is rigid.*

Proof. Recall that η is the unit of the Boardman–Vogt tensor product, hence $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(\eta,P)$ is canonically isomorphic to P for every operad P. This shows that (ii) implies (i). Clearly, (iii) implies (ii). Let us prove that (i) implies (iii). Recall that we have a concrete description of the underlying category of $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(Q,P)$ (see paragraph 5.2.4). If f and g are two objects of this category, then an isomorphism between them is given by isomorphisms α_c in $P(f(c),g(c))$, where c ranges through the colours of Q, such that $g(q)\circ(\alpha_{c_1},\dots,\alpha_{c_n})=\alpha_d\circ f(q)$. Since P is rigid, these isomorphisms have to be the identity. This implies that f has to be equal to g, and that α is an identity, thereby proving the result. $\square$

The sets $\mathcal{I}$, $\mathcal{J}^\flat$ and $\mathcal{J}$ appearing in the remainder of the section are those introduced in paragraph 5.2.5.

Proposition 5.6.2. *An operad is $\mathcal{J}^\flat$-local if and only if it is rigid.*

Proof. Let P be an operad. Denote by $e\colon P\to 1$ the unique map of operads from P to the terminal operad. For any tree T, we have

$$j_T\perp e \iff j\otimes_{\mathcal{O}\mathrm{p}}T\perp e \iff j\perp \underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(T,e),$$

where $f\perp g$ is notation for saying that f has the unique left lifting property with respect to g. Since 1 is the terminal operad, so is $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(T,1)$ and $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(T,e)$ is the unique map from $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(T,P)$ to the terminal operad. This means that P is j_T-local if and only if $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(T,P)$ is j-local, that is, if and only if $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}}(T,P)$ is rigid (see paragraph 5.4.1). The result then follows from Proposition 5.6.1. $\square$

Proposition 5.6.3. *A dendroidal set is $(\mathcal{I}\cup\mathcal{J})$-local if and only if it is the nerve of a rigid operad.*

Proof. Proposition 5.1.12 precisely says that a dendroidal set is $\mathcal{I}$-local if and only if it is the nerve of an operad. The previous proposition shows that such a dendroidal set is $\mathcal{J}$-local if and only if the operad of which it is the nerve is rigid, thereby proving the result. $\square$

Theorem 5.6.4. *The quasi-category* $\mathrm{Aut}(\Omega\text{-}\mathcal{S}p)$ *is a contractible Kan complex.*

Proof. We are going to apply Proposition 3.8 to $A=\Omega$ and $S=\mathcal{I}\cup\mathcal{J}$. Let us check that the hypotheses are fulfilled. Using the previous proposition, this amounts to verifying that

(i) objects of Ω are rigid operads;
(ii) autoequivalences of $\mathcal{O}\mathrm{p}_\mathrm{r}$ restrict to autoequivalences of Ω.

The first point is obvious and the second point is Proposition 5.4.2. We can thus apply the proposition and we get that $\mathrm{Aut}(\Omega\text{-}\mathcal{S}p)$ is a full subcategory of $\mathrm{Aut}(\Omega)$. But $\mathrm{Aut}(\Omega)$ is a contractible groupoid by Theorem 5.5.6 and the result follows. $\square$

6. Autoequivalences of the $(\infty, 1)$-category of non-symmetric ∞-operads

The purpose of this section is to show that the quasi-category of autoequivalences of the quasi-category of planar Ω-spaces, which we use as a model for non-symmetric ∞-operads, is the discrete category $\mathbf{Z}/2\mathbf{Z}$. The combinatorics is similar to the symmetric case, the main differences being due to the fact that a planar tree has no non-trivial automorphisms.

6.1. Preliminaries on non-symmetric operads and planar trees.

6.1.1. We will denote by $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ the category of (small) non-symmetric coloured operads, i.e., operads without an action of the symmetric group. There is a forgetful functor from $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ to the category $\mathcal{C}\mathrm{oll}_{\mathrm{ns}}$ of non-symmetric collections, and this functor admits a left adjoint.

6.1.2. Recall from paragraph 5.1.3 that to every planar tree T, we can associate a non-symmetric collection $K_p(T)$. We define the non-symmetric operad $\Omega_{\mathrm{ns}}(T)$ as the free non-symmetric operad on $K_p(T)$.

6.1.3. The category of planar trees Ω_{ns}, introduced by Moerdijk in [MT10], is defined as follows: the objects of Ω_{ns} are planar trees (up to isomorphism) and the set of morphisms in Ω_{ns} from an object S to an object T is given by

$$\Omega_{\mathrm{ns}}(S, T) = \mathcal{O}\mathrm{p}_{\mathrm{ns}}(\Omega_{\mathrm{ns}}(S), \Omega_{\mathrm{ns}}(T)).$$

By definition, there is a canonical fully faithful functor $\Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns}}$ and we will always consider Ω_{ns} as a full subcategory of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ using this functor.

Remark 6.1.4. One important difference between Ω_{ns} and Ω is that $\mathrm{aut}_{\Omega_{\mathrm{ns}}}(T)$ is trivial for every object T of Ω_{ns}.

6.1.5. As in paragraph 5.1.7, for any planar tree T, we have a category $\mathbb{C}/T$ and a diagram $\mathcal{D}_T \colon \mathbb{C}/T \to \Omega_{\mathrm{ns}}$. A similar proof as the one of Proposition 5.1.8 shows that for every object T of Ω_{ns}, the canonical morphism

$$\operatorname{colim} \mathcal{D}_T = \operatorname*{colim}_{(C, C\to T)\in\mathbb{C}/T} C \longrightarrow T$$

is an isomorphism in Ω_{ns}, and that, moreover, the inclusion functor $\Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns}}$ preserves this colimit.

6.1.6. The category of *planar dendroidal sets* $\mathrm{Pr}(\Omega_{\mathrm{ns}})$ is the category of presheaves on Ω_{ns}. The inclusion $\Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns}}$ induces a *planar dendroidal nerve* functor $N_{\mathrm{ns,d}} \colon \mathcal{O}\mathrm{p}_{\mathrm{ns}} \to \mathrm{Pr}(\Omega_{\mathrm{ns}})$.

6.1.7. The *spine* of a planar tree T is the planar dendroidal set

$$I_T = \operatorname*{colim}_{(C, C\to T)\in\mathbb{C}/T} C,$$

where the colimit is taken in $\mathrm{Pr}(\Omega_{\mathrm{ns}})$. There is a canonical monomorphism of planar dendroidal sets $i_T \colon I_T \to T$. We will denote by $\mathcal{I}$ the set

$$\mathcal{I} = \{i_T \mid T \in \mathrm{Ob}(\Omega_{\mathrm{ns}})\}$$

of spine inclusions.

Let now X be a planar dendroidal set. For any planar tree T, the map i_T induces a *Segal map*

$$X(T) \simeq \mathrm{Pr}(\Omega_{\mathrm{ns}})(T, X) \longrightarrow \mathrm{Pr}(\Omega_{\mathrm{ns}})(I_T, X) \simeq \lim_{(C, C\to T)\in\mathbb{C}/T} X(C).$$

We will say that X *satisfies the Segal condition* if all the Segal maps are bijections. This exactly means that X is $\mathcal{I}$-local.

Proposition 6.1.8 (Cisinski–Moerdijk, Weber)**.** *The planar dendroidal nerve functor is fully faithful. Moreover, its essential image consists of the planar dendroidal sets satisfying the Segal condition.*

Proof. The proof of [CM13a, Corollary 2.6] can be adapted to the case of planar dendroidal sets. It also follows from the general machinery of [Web07]. □

Remark 6.1.9. The first assertion of the previous proposition precisely means that the inclusion functor $\Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns}}$ is dense.

6.2. The quasi-category of complete planar dendroidal Segal spaces.

6.2.1. The category of *planar dendroidal spaces* is the category $\mathrm{sPr}(\Omega_{\mathrm{ns}})$ of simplicial presheaves on Ω_{ns}. We will consider the category of planar dendroidal sets as a full subcategory of the category of planar dendroidal spaces (as in paragraph 5.2.1).

6.2.2. In the definition of the Boardman–Vogt tensor product of operads, it is crucial that the operads under consideration are symmetric. However, the tensor product still makes sense without the symmetries when at least one of the operads involved is a category.

More precisely, the category of non-symmetric operads is tensored over the category of categories. We will denote by $\otimes_{\mathcal{O}\mathrm{p}_{\mathrm{ns}}}$ this tensor. The tensor $\otimes_{\mathcal{O}\mathrm{p}_{\mathrm{ns}}}$ is closed and we will denote by $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}_{\mathrm{ns}}}$ the associated enrichment over categories. This means that if C is a category, and P and Q are non-symmetric operads, then there is a canonical bijection

$$\mathcal{O}\mathrm{p}_{\mathrm{ns}}(C \otimes_{\mathcal{O}\mathrm{p}_{\mathrm{ns}}} P, Q) \simeq \mathcal{C}\mathrm{at}(C, \underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}_{\mathrm{ns}}}(P, Q)).$$

The category $\underline{\mathsf{Hom}}_{\mathcal{O}\mathrm{p}_{\mathrm{ns}}}(P, Q)$ can be described in the same way as the underlying category of the operad of morphisms between operads described in paragraph 5.2.4.

6.2.3. Recall from paragraph 6.1.7 that we denote by $\mathcal{I}$ the set

$$\mathcal{I} = \{i_T \mid T \in \mathrm{Ob}(\Omega_{\mathrm{ns}})\}$$

of spine inclusions.

As in the non-planar case, for every planar tree T, we have a canonical map

$$j_T \colon J \otimes_{\mathcal{O}\mathrm{p}_{\mathrm{ns}}} T \longrightarrow T$$

of non-symmetric operads, where J denotes the contractible groupoid on two objects. We will denote by $\mathcal{J}^\flat$ and $\mathcal{J}$ the sets

$$\mathcal{J}^\flat = \{j_T \mid T \in \mathrm{Ob}(\Omega_{\mathrm{ns}})\} \quad \text{and} \quad \mathcal{J} = \{N_{\mathrm{ns,d}}(j_T) \mid T \in \mathrm{Ob}(\Omega_{\mathrm{ns}})\}.$$

The sets $\mathcal{I}$ and $\mathcal{J}$ will be considered as sets of maps in planar dendroidal sets or planar dendroidal spaces depending on the context.

Definition 6.2.4. The *model category for complete planar dendroidal Segal spaces* or Ω_{ns}*-spaces* is the left Bousfield localization of the injective model structure on planar dendroidal spaces by the set $\mathcal{I} \cup \mathcal{J}$.

Definition 6.2.5. The *quasi-category of* Ω_{ns}*-spaces* is the localization of the quasi-category $\mathcal{P}(\Omega_{\mathrm{ns}})$ by the set $\mathcal{I} \cup \mathcal{J}$. We will denote it by Ω_{ns}-$\mathcal{S}p$.

Proposition 6.2.6. *The quasi-category underlying the model category of Ω_{ns}-spaces is canonically equivalent to the quasi-category of Ω_{ns}-spaces.*

Proof. The proof is the same as the one of Proposition 5.2.9. □

6.3. Autoequivalences of the category of non-symmetric operads.

6.3.1. For each $n \geq 1$, we denote by μ_n the mirror permutation in Σ_n, that is, the permutation defined by

$$\mu_n(i) = n - i + 1, \quad 1 \leq i \leq n.$$

We define an endofunctor M of the category $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ in the following way:

(i) Given an operad P in $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$, the operad $M(P)$ has the same colours as P and, for colours $c_1, \ldots, c_n$ and c of $M(P)$, we set

$$M(P)(c_1, \ldots, c_n; c) = P(c_{\mu_n(1)}, \ldots, c_{\mu_n(n)}; c).$$

(ii) For a map of operads $f\colon P \to Q$ in $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$, the map $M(f)$ is defined on components by

$$M(f)_{(c_1,\ldots,c_n;c)}(p) = f_{(c_{\mu_n(1)},\ldots,c_{\mu_n(n)};c)}(p),$$

where p is an operation in $M(P)(c_1, \ldots, c_n; c)$.

It is easy to check that this functor is indeed well-defined. Obviously, $M \circ M$ is the identity and M is hence an autoequivalence. We will call M the *mirror autoequivalence*. Note that the mirror autoequivalence sends a planar tree to the planar tree obtained by reversing the orientation of the plane.

The autoequivalence M defines a monoid morphism

$$\mathbf{Z}/2\mathbf{Z} \longrightarrow \mathrm{aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns}})$$

and hence a strict monoidal functor

$$(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}} \longrightarrow \mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns}}),$$

where $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}}$ denotes the discrete category on the set $\mathbf{Z}/2\mathbf{Z}$ endowed with the strict monoidal structure given by the group law of $\mathbf{Z}/2\mathbf{Z}$, and the category $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns}})$ is endowed with the strict monoidal structure given by composition of functors.

Proposition 6.3.2. *Let F be an autoequivalence of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$. Then F preserves η, the corollas and the root maps of the corollas.*

Proof. The proofs of Corollary 5.3.3, Proposition 5.3.11 and Proposition 5.3.12 are easily adapted using the following notion of non-symmetric pseudo-corollas: a *non-symmetric pseudo-corolla* is a non-symmetric operad P with a unique non-trivial operation p such that the only colours of P are the inputs and the output of p. Non-symmetric pseudo-corollas can be characterized as in Lemma 5.3.8, and non-symmetric n-corollas can be characterized in terms of non-symmetric pseudo-corollas as in Lemma 5.3.10. □

6.3.3. We will denote by Σ the group $\prod_{n\geq 0} \Sigma_n$. (Note that Σ_0 and Σ_1 are trivial.) We will associate to every autoequivalence F of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ an element $\sigma(F)$ in Σ. Let us define its components $\sigma(F)_n$.

Let thus F be an autoequivalence of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ and let $n \geq 0$. Since F is fully faithful, it induces a bijection $\mathcal{O}\mathrm{p}_{\mathrm{ns}}(\eta, C_n) \to \mathcal{O}\mathrm{p}_{\mathrm{ns}}(F(\eta), F(C_n))$. By the previous proposition, we have $F(\eta) \simeq \eta$ and $F(C_n) \simeq C_n$. Moreover, since η and C_n have no non-trivial automorphisms, these isomorphisms are canonical. We thus obtain an automorphism of the set $\mathcal{O}\mathrm{p}_{\mathrm{ns}}(\eta, C_n)$ and, since by the previous proposition the root map of C_n is preserved, an automorphism of the input colours of C_n, that is, an element in Σ_n. This permutation is by definition the component $\sigma(F)_n$ of $\sigma(F)$.

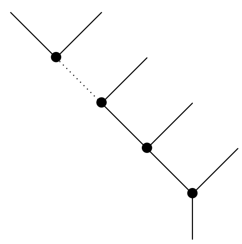
Clearly, if $F = M$ is the mirror autoequivalence, we have $\sigma(M)_n = \mu_n$. In particular, $\sigma(M)_2$ is the transposition τ of Σ_2.

6.3.4. Given any autoequivalence F of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$, we will denote by $\widetilde{F}$ the autoequivalence

$$\widetilde{F} = \begin{cases} F & \text{if } \sigma(F)_2 = 1, \\ M \circ F & \text{if } \sigma(F)_2 = \tau. \end{cases}$$

Note that $\sigma(\widetilde{F})_2 = 1$ for every autoequivalence F.

6.3.5. For every $n \geq 2$, we will denote by B_n the planar binary tree with n leaves of the following shape:



In particular, B_2 is the 2-corolla C_2.

Lemma 6.3.6. *Let F be an autoequivalence of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$. Then $\widetilde{F}(B_n) \simeq B_n$ and $\widetilde{F}$ preserves all maps from η to B_n for every $n \geq 2$.*

Proof. We prove it by induction on $n \geq 2$. If $n = 2$, then B_2 is the 2-corolla C_2 which is preserved by Proposition 6.3.2. Moreover, since $\sigma(\widetilde{F})_2 = 1$, we know that $\widetilde{F}$ preserves all maps from η to $C_2 = B_2$.

Suppose now that the result is true for some $n \geq 2$. We have the following pushout square

$$\begin{tikzcd} \eta \arrow[r] \arrow[d] & B_n \arrow[d, "u"] \\ B_2 \arrow[r, "l"'] & B_{n+1}, \end{tikzcd}$$

where the top map is the root map, the map on the left corresponds to the left leaf of B_2, u is the inclusion of B_n into the upper part of B_{n+1} and l is the inclusion of B_2 into the lower part of B_{n+1}. By induction hypothesis, the maps from η to B_2 and B_n are preserved by $\widetilde{F}$. Since $\widetilde{F}$ preserves pushouts, it has to preserve B_{n+1}. Moreover, since B_{n+1} has no non-trivial automorphisms, it also has to preserve l and u.

Let now $f\colon \eta \to B_{n+1}$ be any morphism. Such a map corresponds to a colour of B_{n+1} which has to belong either to B_2 or to B_n. More precisely, either there exists a map $g\colon \eta \to B_2$ such that $f = lg$, or there exist a map $h\colon \eta \to B_n$ such that $f = uh$. Since l and u are preserved and any map from η to B_2 or B_n is preserved by induction hypothesis, we obtain that $\widetilde{F}$ preserves f. $\square$

Proposition 6.3.7. *If F is an autoequivalence of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$, then $\sigma(\widetilde{F}) = 1$. In other words, $\widetilde{F}$ preserves all maps from η to the n-corolla for every $n \geq 0$.*

Proof. The cases $n = 0$ and $n = 1$ are trivial, and the case $n = 2$ is true by definition of $\widetilde{F}$. Let $n \geq 3$ and consider a map $f\colon \eta \to C_n$. We have a unique map $t\colon C_n \to B_n$ from C_n to B_n. This map corresponds to the total composition of B_n. Set $g = tf$ and consider the commutative triangle

$$\begin{tikzcd} \eta \arrow[r, "g"] \arrow[d, "f"'] & B_n \\ C_n. \arrow[ur, "t"'] & \end{tikzcd}$$

Since t is injective on colours, the map f is the only map making this triangle commute. But the map g and t are both preserved by $\widetilde{F}$ (the first one by Lemma 6.3.6 and the second one by uniqueness). It follows that f is also preserved by $\widetilde{F}$. $\square$

Proposition 6.3.8. *If F is an autoequivalence of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$, then $\widetilde{F}(T) \simeq T$ for every object T of Ω_{ns}. In particular, F sends planar trees to planar trees and thus induces an autoequivalence of Ω_{ns}.*

Proof. Recall from paragraph 6.1.5 that every planar tree T is the colimit of a diagram $\mathcal{D}_T\colon \mathbb{C}/T \to \Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns}}$. By Propositions 6.3.2 and 6.3.7, $\widetilde{F}$ preserves η, C_n and all the maps from $\eta \to C_n$. Moreover, it preserves them up to a canonical isomorphism since these objects do not have non-trivial automorphisms. We thus have a canonical isomorphism $\mathcal{D}_T \simeq \widetilde{F}\mathcal{D}_T$ and hence a chain of canonical isomorphisms

$$T \simeq \operatorname{colim} \mathcal{D}_T \simeq \operatorname{colim} \widetilde{F}\mathcal{D}_T \simeq \widetilde{F}(\operatorname{colim} \mathcal{D}_T) \simeq \widetilde{F}(T),$$

thereby proving the first assertion.

The second assertion follows from the fact that the mirror autoequivalence sends planar trees to planar trees. $\square$

Corollary 6.3.9. *The dense inclusion $\Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns}}$ induces a fully faithful functor* $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns}}) \to \mathrm{Aut}(\Omega_{\mathrm{ns}})$.

Proof. This is immediate from the previous proposition and Proposition 3.7. $\square$

We will show in Section 6.6 that the monoidal category $\mathrm{Aut}(\Omega_{\mathrm{ns}})$ is isomorphic to the discrete monoidal category $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}}$. As a corollary, we will obtain the following theorem:

Theorem. *The functor $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}} \to \mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns}})$ is an equivalence of monoidal categories.*

6.4. Autoequivalences of the category of rigid non-symmetric operads.

6.4.1. A non-symmetric operad is *rigid* if its underlying category is rigid. As in the symmetric case, an operad is rigid if and only if it is j-local, where $j\colon J \to \eta$ is the map j_η of paragraph 6.2.3.

We will denote by $\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$ the full subcategory of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ whose objects are the rigid non-symmetric operads. Note that the non-symmetric operads induced by planar trees are rigid. Hence, as in the symmetric case, we get canonical inclusion functors $\Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$ and $\mathcal{C}\mathrm{at}_{\mathrm{r}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$.

6.4.2. The mirror autoequivalence obviously preserves rigid non-symmetric operads. It thus induces a *mirror autoequivalence* of $\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$, which in turn induces a strict monoidal functor

$$(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}} \longrightarrow \mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}),$$

as in the non-rigid case.

In particular, if F is an autoequivalence of $\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$, we can define an autoequivalence $\widetilde{F}$ of $\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$ as in paragraph 6.3.4.

Proposition 6.4.3. *If F is an autoequivalence of $\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$, then $\widetilde{F}(T) \simeq T$ for every object T of Ω_{ns}. In particular, F sends planar trees to planar trees and thus induces an autoequivalence of Ω_{ns}.*

Proof. The strategy of the proof is basically the same as in the case of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ but taking into account the pertinent modifications for rigid non-symmetric operads as explained in the proof of Proposition 5.4.2 for the case of rigid operads. □

Corollary 6.4.4. *The dense inclusion $\Omega_{\mathrm{ns}} \hookrightarrow \mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$ induces a fully faithful functor $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}) \to \mathrm{Aut}(\Omega_{\mathrm{ns}})$.*

Proof. This is immediate from the previous proposition and Proposition 3.7. □

We will show in Section 6.6 that the monoidal category $\mathrm{Aut}(\Omega_{\mathrm{ns}})$ is isomorphic to the discrete monoidal category $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}}$. As a corollary, we will obtain the following theorem:

Theorem. *The functor $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}} \to \mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns,r}})$ is an equivalence of monoidal categories.*

6.5. Autoequivalences of the category of planar trees.

6.5.1. As already observed, the mirror autoequivalence sends planar trees to planar trees. It thus induces a *mirror autoequivalence* of Ω_{ns} that we will still denote by M, which in turn induces a strict monoidal functor

$$(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}} \longrightarrow \mathrm{Aut}(\Omega_{\mathrm{ns}}),$$

as in the previous two subsections.

In particular, if F is an autoequivalence of Ω_{ns}, we can define an autoequivalence $\widetilde{F}$ of Ω_{ns} as in paragraph 6.3.4.

Proposition 6.5.2. *If F is an autoequivalence of Ω_{ns}, then $\widetilde{F}$ is the identity on objects.*

Proof. The proof of Proposition 5.5.2 can be easily adapted using the canonical isomorphism $\mathcal{D}_T \simeq \widetilde{F}\mathcal{D}_T$ appearing in the proof of Proposition 6.3.8. □

Proposition 6.5.3. *The monoid morphism $\mathbf{Z}/2\mathbf{Z} \to \mathrm{aut}(\Omega_{\mathrm{ns}})$ is an isomorphism.*

Proof. Let F be an autoequivalence of Ω_{ns}. It clearly suffices to show that the autoequivalence $\widetilde{F}$ is the identity. By the previous proposition, we know it is the identity on objects. To prove it is the identity on morphisms, we can reduce, as in the symmetric case, to the case of maps from η to corollas and the result thus follows from Proposition 6.3.7. □

Theorem 6.5.4. *The functor $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}} \to \mathrm{Aut}(\Omega_{\mathrm{ns}})$ is an isomorphism of monoidal categories.*

Proof. The previous proposition states that this functor is bijective on objects. To conclude, it suffices to show that $\mathrm{Aut}(\Omega_{\mathrm{ns}})$ is a discrete category. Let F and G be two autoequivalences of Ω_{ns} and let $\gamma\colon F \to G$ be a natural transformation. For every planar tree T, we have a morphism $\gamma_T\colon F(T) \to G(T)$. If T is the planar tree B_3 described in paragraph 6.3.5, then there are no morphisms $T \to M(T)$ or $M(T) \to T$. This implies that F and G have to be both equal to the identity autoequivalence or to the mirror autoequivalence.

So let F be either the identity autoequivalence or M, and let $\gamma\colon F \to F$ be a natural transformation. Let T be a planar tree and $c\colon \eta \to T$ any colour. Then, by naturality, we have a commutative square

$$\begin{tikzcd} F(\eta) = \eta \arrow[r, "F(c)"] \arrow[d, "\gamma_\eta = 1_\eta"'] & F(T) \arrow[d, "\gamma_T"] \\ F(\eta) = \eta \arrow[r, "F(c)"'] & F(T). \end{tikzcd}$$

This shows that γ_T is the identity on colours and hence the identity map, thereby proving that γ is the identity natural transformation. □

Theorem 6.5.5. *The monoidal categories* $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns}})$ *and* $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns,r}})$ *are equivalent to the discrete monoidal category* $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}}$.

Proof. By Corollaries 6.3.9 and 6.4.4, the categories $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns}})$ and $\mathrm{Aut}(\mathcal{O}\mathrm{p}_{\mathrm{ns,r}})$ are both full (monoidal) subcategories of the category $\mathrm{Aut}(\Omega_{\mathrm{ns}})$. To conclude, it thus suffices to show that every autoequivalence of Ω_{ns} lifts to autoequivalences of $\mathcal{O}\mathrm{p}_{\mathrm{ns}}$ and $\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$. This is obvious since, by Proposition 6.5.3, the autoequivalences of Ω_{ns} are the identity and the mirror autoequivalence. □

6.6. Autoequivalences of the quasi-category of non-symmetric ∞-operads.

6.6.1. The autoequivalence M of Ω_{ns} extends formally to an autoequivalence of the quasi-category $\mathcal{P}(\Omega_{\mathrm{ns}})$. It is easy to see that the sets $\mathcal{I}$ and $\mathcal{J}$ of paragraph 6.2.3 are stable under this autoequivalence and we thus get an induced autoequivalence M of $\Omega_{\mathrm{ns}}\text{-}\mathcal{S}p = (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{P}(\Omega_{\mathrm{ns}})$.

The sets $\mathcal{I}$, $\mathcal{J}^\flat$ and $\mathcal{J}$ appearing in the remainder of the section are those introduced in paragraph 6.2.3.

Proposition 6.6.2. *A non-symmetric operad is* $\mathcal{J}^\flat$*-local if and only if it is rigid.*

Proof. The proof is a trivial adaptation of the proof of Proposition 5.6.2 □

Proposition 6.6.3. *A planar dendroidal set is* $(\mathcal{I} \cup \mathcal{J})$*-local if and only if it is the nerve of a rigid non-symmetric operad.*

Proof. This follows from the previous proposition and Proposition 6.1.8 as in the proof of Proposition 5.6.3. □

Theorem 6.6.4. *The quasi-category* $\mathrm{Aut}(\Omega_{\mathrm{ns}}\text{-}\mathcal{S}p)$ *is canonically equivalent to the discrete category* $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}}$.

Proof. We are going to apply Proposition 3.8 to $A = \Omega_{\mathrm{ns}}$ and $S = \mathcal{I} \cup \mathcal{J}$. Let us check that the hypotheses are fulfilled. Using the previous proposition, this amounts to verifying that

(i) objects of Ω_{ns} are rigid non-symmetric operads;
(ii) autoequivalences of $\mathcal{O}\mathrm{p}_{\mathrm{ns,r}}$ restrict to autoequivalences of Ω_{ns}.

The first point is obvious and the second point is Proposition 6.4.3. We can thus apply the proposition and we get that $\mathrm{Aut}(\Omega_{\mathrm{ns}}\text{-}\mathcal{S}p)$ is a full subcategory of $\mathrm{Aut}(\Omega_{\mathrm{ns}})$. But $\mathrm{Aut}(\Omega_{\mathrm{ns}})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}}$ by Theorem 6.5.4. To conclude, it thus suffices to show that every autoequivalence of Ω_{ns} lifts to an autoequivalence of $\Omega_{\mathrm{ns}}\text{-}\mathcal{S}p$. This follows from paragraph 6.6.1. $\square$

Remark 6.6.5. Using the monoidal structures described in Remark 4.6.5, one can show that $\mathrm{Aut}(\Omega_{\mathrm{ns}}\text{-}\mathcal{S}p)$ and $(\mathbf{Z}/2\mathbf{Z})_{\mathrm{disc}}$ are equivalent as monoidal quasi-categories.

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